# International School
# “Toric topology, combinatorics and data analysis”

International Laboratory of Algebraic Topology and Its Applications,
Faculty of Computer Science, HSE University, Moscow
and
Steklov Mathematical Institute of Russian Academy of Sciences, Moscow
Steklov International Mathematical Center, Moscow
and
Leonhard Euler International Mathematical Institute in Saint Petersburg

EIMI, 3-9 October 2022

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# Program and abstracts

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# Program of the School

## Monday, 3 October:

9:00-10:00 Registration
10:00-11:00 T. E. Panov, “Double cohomology of moment-angle complexes”
11:00-11:30 Coffee break
11:30-12:30 V. M. Buchstaber, “Theory and applications of $n$-valued groups”
12:30-13:30 Short talk/Poster session
13:30-15:00 Lunch
15:00-16:00 A. A. Ayzenberg, “Evasive oracles and homology of moment-angle complexes I”
16:00-16:30 Coffee break
16:30-17:30 A. A. Gaifullin, “Minimal triangulations of manifolds which are like projective planes”
17:30-18:30 Short talk/Poster session
19:00-22:00 Welcome reception

## Tuesday, 4 October:

10:00-11:00 A. Y. Perepechko, “Common refinements of simplicial complexes and Oda’s strong factorization conjecture”
11:00-11:30 Coffee break
11:30-12:30 A. A. Ayzenberg, “Evasive oracles and homology of moment-angle complexes II”
12:30-13:30 Short talk/Poster session
13:30-15:00 Lunch
15:00-17:00 “Round Table”: Big data, algorithms and applications
17:00-17:30 Coffee break
17:30-18:30 Short talk/Poster session

## Wednesday, 5 October:

10:00-18:00 Excursion (bus trip to Petergof)
19:00-22:00 Reception

## Thursday, 6 October:

10:00-11:00 S. Terzić, “The theory of $(2n, k)$ - manifolds via Grassmann and flag manifolds”
11:00-11:30 Coffee break
11:30-12:30 N. Y. Erokhovets, “Combinatorics and hyperbolic geometry of families of three-dimensional polyhedra”
12:30-13:30 Short talk/Poster session

13:30-15:00 Lunch
15:00-17:00 “Round Table”: Toric topology and its interactions with related subjects
17:00-17:30 Coffee break
17:30-18:30 Short talk/Poster session

## Friday, 7 October:

10:00-11:00 A. Y. Vesnin, “Volumes of hyperbolic polyhedra”
11:00-11:30 Coffee break
11:30-12:30 N. Y. Erokhovets, “Toric topology of families of polyhedra”
12:30-13:30 Short talk/Poster session
13:30-15:00 Lunch
15:00-16:00 G. Y. Panina, “Euler class: geometry and combinatorics I”
16:00-16:30 Coffee break
16:30-17:30 R. T. Živaljević, “Bier spheres, generalized moment-angle complexes, and the simplicial Steinitz problem
17:30-18:30 Short talk/Poster session

## Saturday, 8 October:

10:00-11:00 G. Y. Panina, “Euler class: geometry and combinatorics II”
11:00-11:30 Coffee break
11:30-12:30 A. Y. Vesnin, “Knots and links in spatial graph”
12:30-13:30 Short talk/Poster session
13:30-15:00 Lunch
15:00-16:00 F. V. Petrov, “Algebraic and topological methods in combinatorics”
16:00-16:30 Coffee break
16:30-17:30 R. T. Živaljević, “Generalized Tonnetz and discrete Abel-Jacobi map”
17:30-18:30 Short talk/Poster session

## Sunday, 9 October:

10:00-11:00 Short talk/Poster session
11:00-11:30 Coffee break
11:30-12:30 Short talk/Poster session

# Abstracts of the lectures

# A. A. Ayzenberg

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## Evasive oracles and homology of moment-angle complexes

In order to check some property of a graph, one needs to ask a number of questions about edges of a graph. Let n be the number of vertices, so that $m = n(n-1)/2$ is the maximal possible number of edges. The original Aanderaa-Rosenberg conjecture (now proved) states that there exists $a > 0$ such that at least $a \cdot m$ questions are needed to check any monotonic invariant property. A stronger evasiveness conjecture (otherwise called Aanderaa-Karp-Rosenberg conjecture) asserts that exactly m questions are always needed to check a monotonic invariant property. There was much topological research around this stronger statement, relating the subject to the study of fixed point sets of finite group actions on cell complexes.

Instead, I replace a boolean oracle with an oracle operating on real/complex numbers, and, via results of Bjorner-Lovasz, relate the study of evasiveness to the theory of moment-angle complexes known in toric topology. Toral rank conjecture, proved by Ustinovskii for moment-angle complexes, allows to deduce a version of the original Aanderaa-Rosenberg conjecture for non-invariant monotonic properties.

This is quite a new perspective with lots of directions of research: in toric topology, theoretical informatics, and probably even artificial intelligence.

### Prerequisites

It is assumed that the audience knows the definitions of a simplicial complex and that of homology (e.g. simplicial homology). Other concepts needed for understanding will be introduced in the lectures.

# V. M. Buchstaber

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## Theory and applications of $n$-valued groups

We will introduce the main notions and constructions of the $n$-valued groups theory. We will discuss key examples and topical problems of this theory. Results of the $n$-valued groups theory, which have found applications in various areas of mathematics, will be presented.

### Bibliography


V. M. Buchstaber, "$n$-valued groups: theory and applications", Moscow Math. J., 6:1 (2006), 57-84;

V. M. Buchstaber, A. P. Veselov, A. A. Gaifullin, "Classification of involutive commutative two-valued groups", Uspekhi Mat. Nauk, 77:4(466) (2022), 91-172.



# N. Y. Erokhovets

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## Combinatorics, Lobachevsky geometry and toric topology of families of three-dimensional polyhedra

### Combinatorics and hyperbolic geometry of families of three-dimensional polyhedra

It is planned to talk about families of three-dimensional polytopes defined by the condition of cyclic $k$-edge connectivity. Such families include flag polyhedra and Pogorelov polyhedra. Using E.M. Andreev's theorem, we will show that flag polyhedra are realized as bounded polyhedra with the same dihedral angles, while Pogorelov polyhedra are realized as right dihedral angles.

Among the Pogorelov polyhedra there is an important subfamily of fullerenes of simple three-dimensional polyhedra with only pentagonal and hexagonal faces. Such polyhedra are mathematical models of fullerenes, molecular compounds of carbon atoms, which are fundamental objects of quantum physics, quantum chemistry and nanotechnology.

Our focus will be on another family of polytopes, the ideal right-angled polytopes, which play a central role in the Koebe-Andreev-Thurston theorem that every three-dimensional polytope can be realized in Euclidean space so that all its edges touch the three-dimensional sphere. For each family, we will show how to build it from several initial polyhedra using vertex and edge cut operations.

## Toric topology of families of polyhedra

To each simple polytope in the toric topology there is associated a smooth manifold with a torus action, such that the polytope is the orbit space of this action. The topology of this manifold and action depends only on the combinatorics of the polyhedron.

It is planned to discuss the issues of cohomological rigidity: when a polyhedron is uniquely determined by the graded cohomology ring of the moment-angle of the manifold. In particular, we will discuss why Pogorelov polytopes and ideal right-angled polyhedra are rigid. To do this, we describe the cohomology ring of the moment-angle manifold using the combinatorics of a polyhedron, in particular, we show which cohomology classes are rigid, that is, they pass into each other under isomorphism of cohomology rings.


# A. A. Gaifullin

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## Minimal triangulations of manifolds which are like projective planes

In 1987 Brehm and Kühnel proved the following estimate: Any combinatorial triangulation of a closed $d$-manifold which is not homeomorphic to the sphere has at least $3d/2+3$ vertices. Triangulated manifolds that have exactly $3d/2+3$ vertices and are not spheres are interesting combinatorial objects. They possess a lot of nice combinatorial and topological properties. In particular, such manifolds may exist in dimensions 2, 4, 8, and 16 only, and are manifolds 'like projective planes' in the sense of Eells and Kuiper. Until recently, there were only 5 known examples of such triangulated manifolds, namely, the 6-vertex triangulation of the real projective plane, the 9-vertex triangulation of the complex projective plane, and three 15-vertex triangulations of the quaternionic projective plane. Very recently, the speaker has succeeded to construct many (more than $10^{103}$) such triangulations in dimension 16. The course will be devoted to constructions and properties of these combinatorial manifolds.


# G. Y. Panina

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## Euler class: geometry and combinatorics

Theoretically, Euler class is a topological invariant of a fiber bundle. Practically, it is a usefull tool for solving problems where usual continuity tricks do not work.

We will discuss two subjects:

– how to use Euler class in practice. As an example, we shall prove Borsuk-Ulam theorem via Euler class.

– Euler class in the triangulated world: a local combinatorial formula.

# T. E. Panov

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## Double cohomology of moment-angle complexes

There is a cochain complex structure $CH^*(Z_K)$ on the cohomology of a moment-angle complex $Z_K$, obtained by defining a new differential d' on the Hochster decomposition of the Tor-algebra of the face ring of a simplicial complex K. Cohomology of $CH^*(Z_K)$ is called the double cohomology, $HH^*(Z_K)$.

It can be identified with the second double cohomology of a bicomplex obtained by adding the second differential d' to the Koszul differential graded algebra of the face ring of K.

The motivation for defining $HH^*(Z_K)$ comes from persistent cohomology. The double cohomology and the corresponding bigraded barcodes possess a stability property, unlike the ordinary cohomology $H^*(Z_K)$.

# A. Y. Perepechko

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## Common refinements of simplicial complexes and Oda's strong factorization conjecture

Tadao Oda's conjecture states that any proper toric birational map between complete smooth toric varieties can be decomposed into a sequence of blowups with nonsingular invariant centers followed by a sequence of inverses of such maps.

It is expressed combinatorially as follows: given two nonsingular fans of polyhedral cones with the same support, there exists a third fan that can be reached from both fans by sequences of smooth star subdivisions.

We will study this conjecture and a possible approach by Sergio Da Silva and Kalle Karu in dimension 3, which is reduced to studying subdivisions of a single triangle.

# F. V. Petrov

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## Algebraic and topological methods in combinatorics

Both polynomial algebra and algebraic topology are successfully used for proving combinatorial results (usually of existence theorems type) for a while. In a joint work with Roman Karasev (2012) we established an unexpected relation between these methods, which is still not understood satisfactory. I want to discuss both achievements and questions in this area.

# S. Terzić

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## The theory of $(2n, k)$ - manifolds via Grassmann and flag manifolds

We present the basic facts on the theory of $(2n, k)$-manifolds which has been recently developed in our joint works with Victor M. Buchstaber. Throughout the presentation we will also demonstrate these facts in the case of the canonical compact torus action on the complex Grassmann and flag manifolds. Eventually, this will lead to the description of the corresponding equivariant structures and orbit spaces of the complex Grassmann and flag manifolds.

### Bibliography

V. M. Buchstaber and S. Terzić, *Topology and geometry of the canonical action of $T^4$ on the complex Grassmannian $G_{4,2}$ and the complex projective space $CP^5$*, Moscow Math. Jour. Vol. 16, Issue 2, (2016), 237–273.

V. M. Buchstaber and S. Terzić, *Toric Topology of the Complex Grassmann Manifolds*, Moscow Math. **19**, no. 3, (2019) 397-463.

V. M. Buchstaber and S. Terzić, *The foundations of $(2n, k)$-manifolds*, Sb. Math. 210, No. 4, 508-549 (2019).

Victor M. Buchstaber and Svjetlana Terzić, *A resolution of singularities for the orbit spaces $G_{n,2}/T^n$*, Proc. of Steklov Inst. of Math., vol. 317, in press.

# A. Y. Vesnin

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## Volumes of hyperbolic polyhedra

We will consider a class of polyhedra which can be realized with all dihedral angles $\pi/2$ in a Lobachevsky 3-space. These polyhedra are referred as right-angled hyperbolic. It is known that right-angled hyperbolic polyhedra can be used as building blocks for a wide class of hyperbolic 3-manifolds, including some link complements [1]. Following [2] and [3], we will present volume computations and bounds of volumes of right-angled polyhedra in terms of number of vertices. These volumes computations suggested the Right-angled knots conjecture formulated in [4].

### Bibliography

[1] A.Vesnin, Right-angled polyhedra and hyperbolic 3-manifolds, Russian Math. Surveys, 2017, 72(2), 335-374. http://dx.doi.org/10.1070/RM9762

[2] A. Vesnin, A. Egorov, Ideal right-angled polyhedra in Lobachevsky space, Chebyshevskii Sbornik 2020, vol. 21, no. 2, pp. 65—83. http://doi.org/10.22405/2226-8383-2020-21-2-65-83.

[3] S. Alexandrov, N. Bogachev, A. Egorov, A. Vesnin, On volumes of hyperbolic right-angled polyhedra, accepted. Preprint version is available at arXiv:2111.08789.

[4] A Champanerkar, I. Kofman, J. Purcell, Right-angled polyhedra and alternating link, Algebraic and Geometric Topology, 2022, 22, 739-784

## Knots and links in spatial graph

An embedding $f$ of a finite graph $G$ into the 3-sphere is called a spatial embedding of $G$, and $f(G)$ is called as spatial graphs of $G$. The fundamental problem in the theory of spatial graphs is, for any given graph $G$, to classify up to isotopy of the embeddings of $G$.

Firstly, we will discuss intrinsically linked graphs. Any set of pairwise disjoint cycles in $G$ will give a link in $f(G)$. If a spatial graph $f(G)$ contains a non-trivial link, then we say that $f(G)$ is linked. A graph is said to be intrinsically linked if a spatial graph $f(G)$ is linked for any $f$. In 1980's Conway – Gordon and Sachs independently proved that $K_6$ is intrinsically linked, where $K_n$ is the complete graph on $n$ vertices. We will discuss this result and its various generalizations.

Secondly, we will discuss the Yamada polynomial which is useful to determine that two spatial graphs are not equivalent.

# R. T. Živaljević

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## Bier spheres, generalized moment-angle complexes, and the simplicial Steinitz problem

The problem of deciding if a given triangulation of a sphere is realizable as the boundary sphere of a simplicial, convex polytope is known as the "Simplicial Steinitz problem". This is an example of a problem of geometric combinatorics which links together areas of mathematics as distant as combinatorial optimization, toric topology, convex polytopes, algebraic geometry, topological combinatorics, discrete and computational geometry, etc.

It is known (by indirect and non-constructive arguments) that a vast majority of triangulated spheres are "non-polytopal", in the sense that they are not combinatorially isomorphic to the boundary of a convex polytope. This holds, in particular, for Bier spheres Bier(K) (named after T. Bier), the $(n-2)$-dimensional, combinatorial spheres on $2n$-vertices, constructed with the aid of simplicial complexes K on $n$ vertices.

Emphasizing connections with toric topology (generalized moment-angle complexes), we will review "hidden geometry" of Bier spheres by describing their natural geometric realizations, compute their volume, describe an effective criterion for their polytopality, and associate to Bier(K) a natural coarsening Fan(K) of the Braid fan. We also establish a connection of Bier spheres of maximal volume with recent generalizations of the classical Van Kampen-Flores theorem and clarify the role of Bier spheres in the theory of generalized permutohedra.

## Generalized Tonnetz and discrete Abel-Jacobi map

Motivated by the classical *Tonnetz*, a conceptual lattice diagram representing tonal space, described originally by Leonhard Euler in 1739, we introduce and study the combinatorics and topology of more general simplicial complexes $Tonn^{n,k}(L)$ of *Tonnetz type*. Out main result is that for a sufficiently generic choice of parameters the generalized tonnetz $Tonn^{n,k}(L)$ is a triangulation of a $(k-1)$-dimensional torus $T^{k-1}$. In the proof we construct and use the properties of a *discrete Abel-Jacobi map*, which takes values in the torus $T^{k-1} \cong \mathbb{R}^{k-1}/\Lambda$ where $\Lambda \cong \mathbb{A}^*_{k-1}$ is the permutohedral lattice.

## Bibliography

Euler, Leonhard (1739). Tentamen novae theoriae musicae ex certissismis harmoniae principiis dilucide expositae. Saint Petersburg Academy. p. 147.

F. D. Jevti, R. T. Živaljević, Generalized Tonnetz and discrete Abel-Jacobi map, Topological Methods in Nonlinear Analysis, Vol 57, No 2 (2021).