ON THE CRYPTOMORPHISM BETWEEN DAVIS' SUBSET LATTICES, ATOMIC LATTICES, AND CLOSURE SYSTEMS UNDER T1 SEPARATION AXIOM

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NTR webinar
September 15, 2022

MMCP
Models and Methods
of Computational Pragmatics

OUTLINE

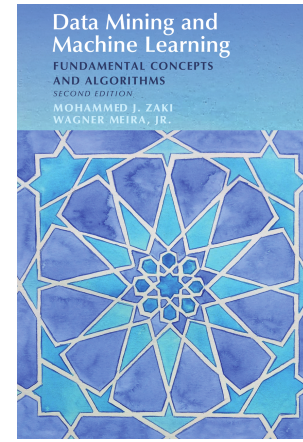
https://arxiv.org/abs/2209.12256

- Motivation from Data Mining perspective
- Closure systems and concept lattices
  - T1 closure systems
- The lattice of all atomic lattices L_n
- Davis' subset lattices
- Enumeration
  - Two cases – two algorithms
  - New numbers and sequences for OEIS
- Extra results
  - Kleitman's maximal union-free families for n=6
  - Standard context representation of L_n
  - Extremal lattices and the breadth of atomic lattices
  - Upper bounds for $|L_n|$

MOTIVATION FROM DATA MINING

The FIMI’03 best implementation award was granted to Gosta Grahne and Jianfei Z
(on the left). The award consisted of the most frequent itemset: *{diapers, beer}*.

MOTIVATION FROM DATA MINING



- Frequent itemset mining (Zaki & Meira, 2022)

D	*A*	*B*	*C*	*D*	*E*
1	1	1	0	1	1
2	0	1	1	0	1
3	1	1	0	1	1
4	1	1	1	0	1
5	1	1	1	1	1
6	0	1	1	1	0

(a) Binary database

t	**i**(*t*)
1	*ABDE*
2	*BCE*
3	*ABDE*
4	*ABCE*
5	*ABCDE*
6	*BCD*

(b) Transaction database

x	*A*	*B*	*C*	*D*	*E*
t(*x*)	1	1	2	1	1
	3	2	4	3	2
	4	3	5	5	3
	5	4	6	6	4
		5			5
		6			

(c) Vertical database

MOTIVATION FROM DATA MINING
(ZAKI & MEIRA, 2022)

Tid	Itemset
1	$ABDE$
2	BCE
3	$ABDE$
4	$ABCE$
5	$ABCDE$
6	BCD

(a) Transaction database

sup	Itemsets
6	B
5	E, BE
4	$A, C, D, AB, AE, BC, BD, ABE$
3	$AD, CE, DE, ABD, ADE, BCE, BDE, ABDE$

(b) Frequent itemsets ($minsup = 3$)

MOTIVATION FROM DATA MINING
(ZAKI & MEIRA, 2022)

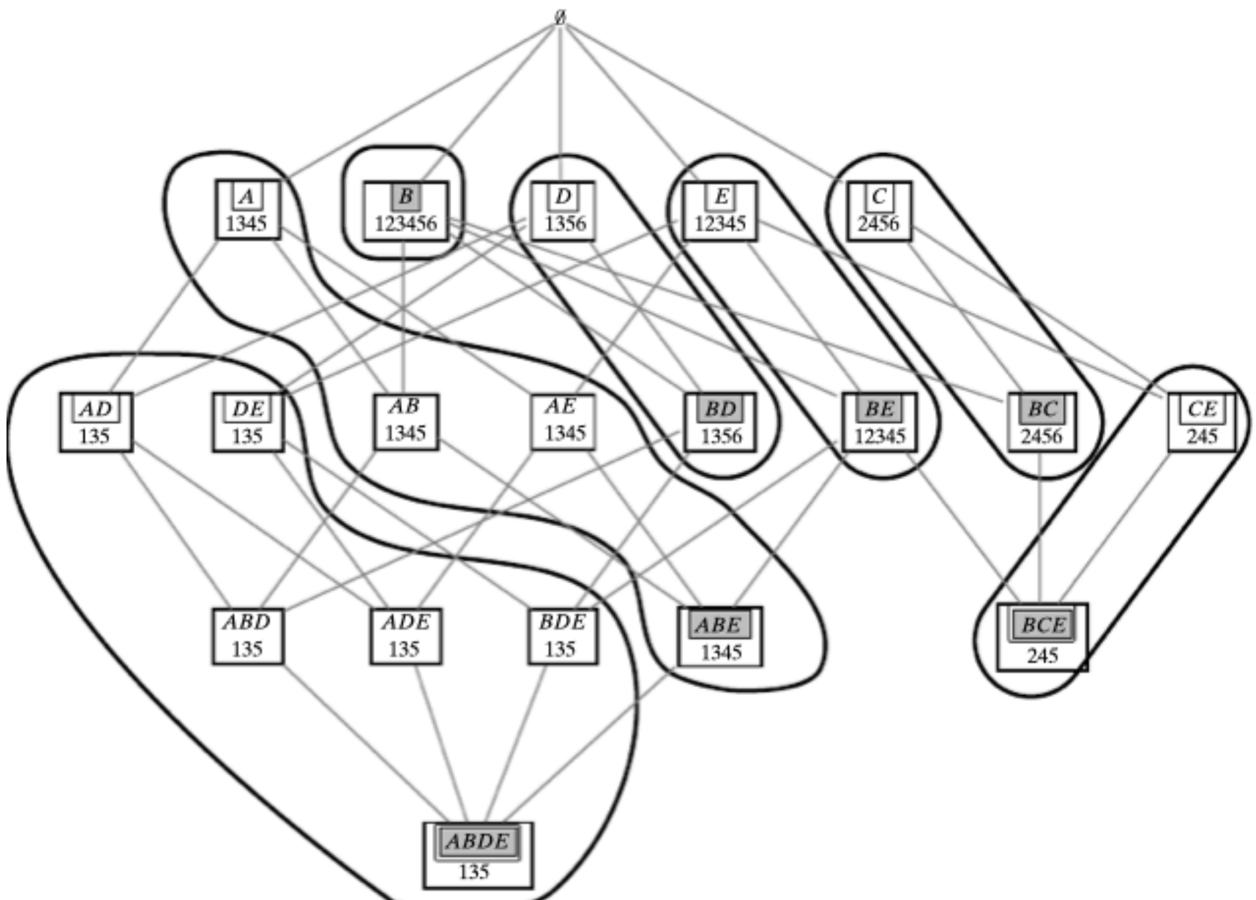


Figure 9.2. Frequent, closed, minimal generators, and maximal frequent itemsets. Itemsets that are boxed and shaded are closed, whereas those within boxes (but unshaded) are the minimal generators; maximal itemsets are shown boxed with double lines.

MOTIVATION FROM DATA MINING
(ZAKI & MEIRA, 2022)

9.5 FURTHER READING

The concept of closed itemsets is based on the elegant lattice theoretic framework of formal concept analysis in Ganter, Wille, and Franzke (1997). The Charm algorithm for mining frequent closed itemsets appears in Zaki and Hsiao (2005), and the GenMax method for mining maximal frequent itemsets is described in Gouda and Zaki (2005). For an Apriori style algorithm for maximal patterns, called MaxMiner, that uses very effective support lower bound based itemset pruning see Bayardo Jr (1998). The notion of minimal generators was proposed in Bastide et al. (2000); they refer to them as *key patterns*. Nonderivable itemset mining task was introduced in Calders and Goethals (2007).

Bastide, Y., Taouil, R., Pasquier, N., Stumme, G., and Lakhal, L. (2000). Mining frequent patterns with counting inference. *ACM SIGKDD Explorations Newsletter*, 2 (2), 66–75.

Bayardo Jr, R. J. (1998). Efficiently mining long patterns from databases. *Proceedings of the ACM SIGMOD International Conference on Management of Data*. ACM, pp. 85–93.

Calders, T. and Goethals, B. (2007). Non-derivable itemset mining. *Data Mining and Knowledge Discovery*, 14 (1), 171–206.

Ganter, B., Wille, R., and Franzke, C. (1997). *Formal Concept Analysis: Mathematical Foundations*. New York: Springer-Verlag.

CLOSURE SYSTEMS AND LATTICES
(COLOMB, IRLANDE, AND RAYNAUD, 2010)

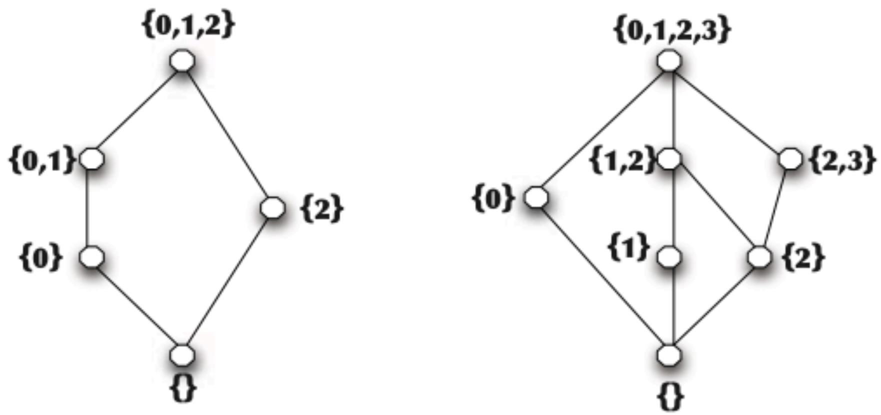


Fig. 1. Two moore families : On the left, family $\{\emptyset, \{0\}, \{0,1\}, \{2\}, \{0,1,2\}\}$ on the set $\{0,1,2\}$. On the right, family $\{\emptyset, \{0\}, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{0,1,2,3\}\}$ on the set $\{0,1,2,3\}$.

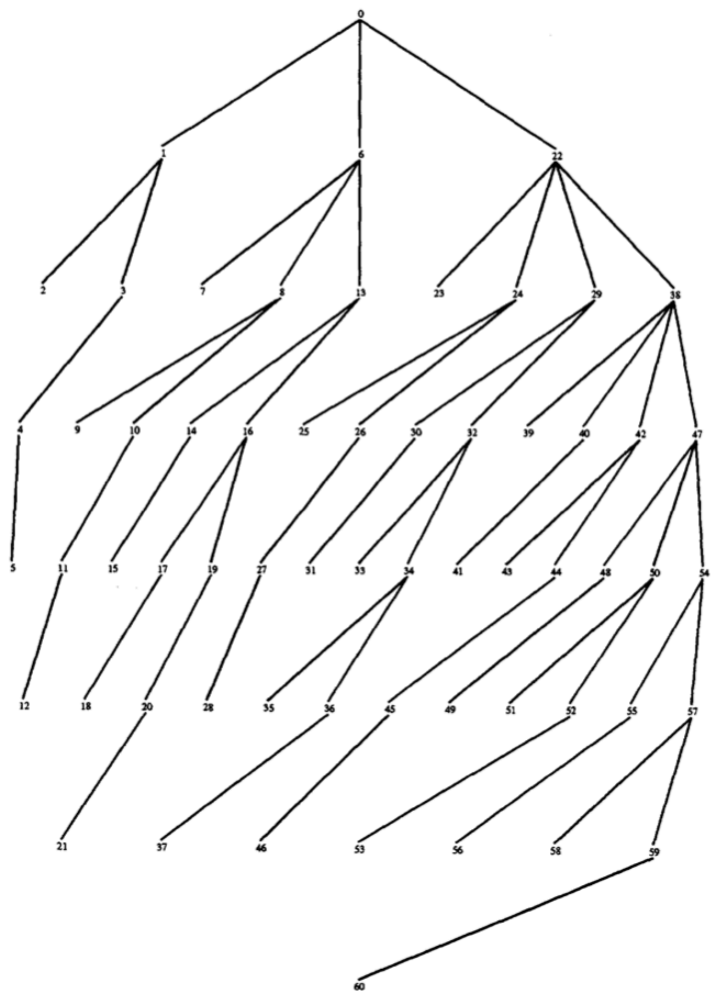
CLOSURE SYSTEMS AND LATTICES

(COLOMB, IRLANDE, AND RAYNAUD, 2010)

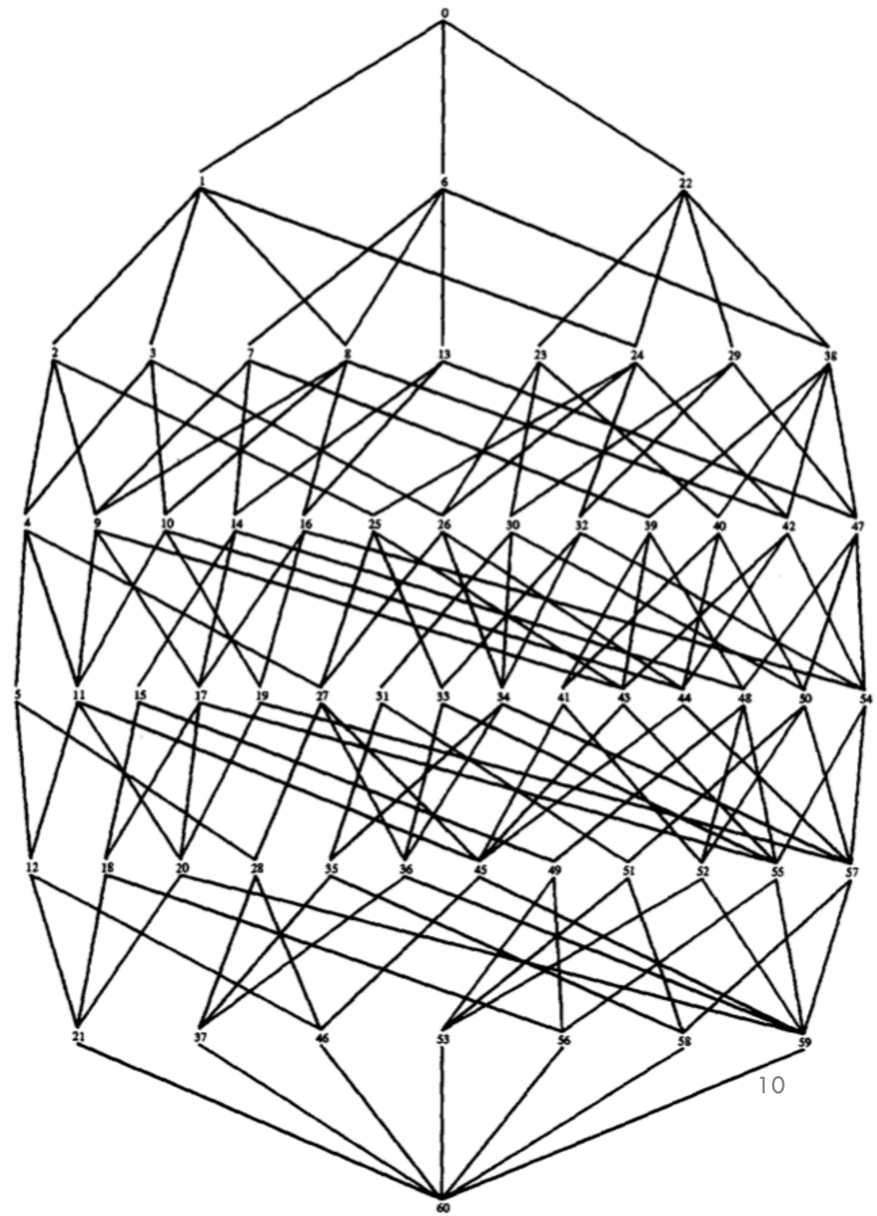
Table 1. Known values of $|\mathcal{M}_n|$ on $n \leq 7$

n	$\|\mathtt{M}_n\|$	Référence
0	1	
1	2	
2	7	
3	61	
4	2 480	
5	1 385 552	[11]
6	75 973 751 474	[1]
7	14 087 648 235 707 352 472	This paper

THE LATTICE OF ALL MOORE FAMILIES FOR n=3



A. Higuchi / Discrete Mathematics 179 (1998) 267–272



OEIS SEQUENCE A102896

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Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A102896 Number of ACI algebras (or semilattices) on n generators with no annihilator. 33

1, 2, 7, 61, 2480, 1385552, 75973751474, 14087648235707352472 (list; graph; refs; listen; history; edit; text; internal format)

OFFSET 0,2

COMMENTS Or, number of Moore families on an n-set, that is, families of subsets that contain the universal set {1,...,n} and are closed under intersection.
Or, number of closure operators on a set of n elements.
An ACI algebra or semilattice is a system with a single binary, idempotent, commutative and associative operation.
Also the number of set-systems on n vertices that are closed under union. The BII-numbers of these set-systems are given by A326875. - Gus Wiseman, Jul 31 2019

REFERENCES
G. Birkhoff, Lattice Theory. American Mathematical Society, Colloquium Publications, Vol. 25, 3rd ed., Providence, RI, 1967.
Maria Paola Bonacina and Nachum Dershowitz, Canonical Inference for Implicational Systems, in Automated Reasoning, Lecture Notes in Computer Science, Volume 5195/2008, Springer-Verlag.
P. Colomb, A. Irlande and O. Raynaud, Counting of Moore Families for n=7, International Conference on Formal Concept Analysis (2010). [From Pierre Colomb (pierre(AT)colomb.me), Sep 04 2010]
E. H. Moore, Introduction to a Form of General Analysis, AMS Colloquium Publication 2 (1910), pp. 53-80.

OEIS OEIS SEQUENCE A102896

EXAMPLE	From Gus Wiseman, Jul 31 2019: (Start) The a(0) = 1 through a(2) = 7 set-systems closed under union: {} {} {} {{1}} {{1}} {{2}} {{1,2}} {{1},{1,2}} {{2},{1,2}} {{1},{2},{1,2}} (End)
MATHEMATICA	Table[Length[Select[Subsets[Subsets[Range[n], {1, n}]], SubsetQ[#, Union@@@Tuples[#, 2]]&]], {n, 0, 3}] (* Gus Wiseman, Jul 31 2019 *)
CROSSREFS	For set-systems closed under union: - The covering case is A102894. - The unlabeled case is A193674. - The case also closed under intersection is A306445. - Set-systems closed under overlapping union are A326866. - The BII-numbers of these set-systems are given by A326875. Cf. A102895, A102897, A108798, A108800, A193675, A000798, A014466, A326878, A326880, A326881. Sequence in context: A046846 A111010 A089307 * A088107 A132524 A153694 Adjacent sequences: A102893 A102894 A102895 * A102897 A102898 A102899
KEYWORD	nonn,hard,more
AUTHOR	Mitch Harris, Jan 18 2005
EXTENSIONS	N. J. A. Sloane added a(6) from the Habib et al. reference, May 26 2005 Additional comments from Don Knuth, Jul 01 2005 a(7) from Pierre Colomb (pierre(AT)colomb.me), Sep 04 2010
STATUS	approved

WHAT DOES DON KNUTH SAY?

In actual fact, I was tearing my hair out for awhile, because I couldn't believe that this would be so complicated. Maybe some day I'll learn the right way to tackle this problem.

https://www-cs-faculty.stanford.edu/~knuth/programs/horn-count.w

WHAT DOES DON KNUTH WRITE?

Table 3

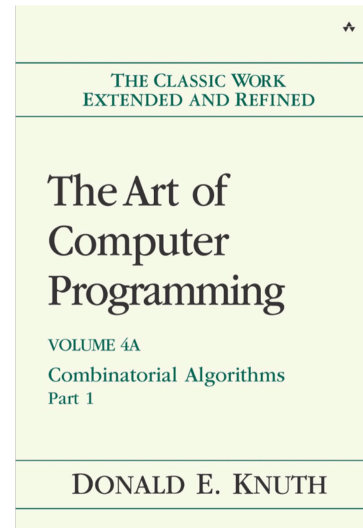
BOOLEAN FUNCTIONS OF n VARIABLES

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
arbitrary	2	4	16	256	65,536	4,294,967,296	18,446,744,073,709,551,616
self-dual	0	2	4	16	256	65,536	4,294,967,296
monotone	2	3	6	20	168	7,581	7,828,354
both	0	1	2	4	12	81	2,646
Horn	2	4	14	122	4,960	2,771,104	151,947,502,948
Krom	2	4	16	166	4,170	224,716	24,445,368
threshold	2	4	14	104	1,882	94,572	15,028,134
symmetric	2	4	8	16	32	64	128
canalizing	2	4	14	120	3,514	1,292,276	103,071,426,294

Table 4

BOOLEAN FUNCTIONS DISTINCT UNDER PERMUTATION OF VARIABLES

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
arbitrary	2	4	12	80	3,984	37,333,248	25,626,412,338,274,304
self-dual	0	2	2	8	32	1,088	6,385,408
monotone	2	3	5	10	30	210	16,353
both	0	1	1	2	3	7	30
Horn	2	4	10	38	368	29,328	216,591,692
Krom	2	4	12	48	308	3,028	49,490
threshold	2	4	10	34	178	1,720	590,440
canalizing	2	4	10	38	294	15,774	149,325,022



KNOWN ASYMPTOTIC AND BOUNDS
(V.B. ALEKSEEV, 1989)

В данной работе, в частности, доказывается следующая теорема.

Теорема 1. *Для числа $\alpha(n)$ семейств подмножеств n-элементного множества, замкнутых относительно пересечений, справедливо асимптотическое*

равенство

$$\log_2 \alpha(n) \sim C_n^{[n/2]} \quad \text{при} \quad n \to \infty. \qquad (2)$$

Более точно, $\log_2 \alpha(n) = C_n^{[n/2]}(1 + O(n^{-1/4} \log_2 n))$.

Задача о числе семейств подмножеств, замкнутых относительно пересечений, рассматривалась в [1], где для $\log_2 \alpha(n)$ получены оценки

$$C_n^{[n/2]} \leqslant \log_2 \alpha(n) \leqslant C_n^{[n/2]} \log_2 n (1 + o(1)) \quad \text{при} \quad n \to \infty.$$

Позднее теми же авторами и Д. Клейтменом получена оценка

$$\log_2 \alpha(n) \leqslant C_n^{[n/2]} 2\sqrt{2}(1 + o(1)),$$

т. е. найден порядок для $\log_2 \alpha(n)$ при $n \to \infty$.

T1 CLOSURE SYSTEMS

A334254 Number of closure operators on a set of n elements which satisfy the T_1 separation axiom. 4

1, 2, 1, 8, 545, 702525, 66960965307 (list; graph; refs; listen; history; edit; text; internal format)

OFFSET 0,2

COMMENTS The T_1 axiom states that all singleton sets {x} are closed.
For n>1, this property implies strictness (meaning that the empty set is closed).

LINKS Table of n, a(n) for n=0..6.
Dmitry I. Ignatov, Supporting iPython code for counting closure systems w.r.t. the T_1 separation axiom, Github repository
Dmitry I. Ignatov, PDF of the supporting iPython notebook
Eric Weisstein's World of Mathematics, Separation Axioms
Wikipedia, Separation Axiom

EXAMPLE The a(3) = 8 set-systems of closed sets:
{{1,2,3},{1},{2},{3},{}}
{{1,2,3},{1,2},{1},{2},{3},{}}
{{1,2,3},{1,3},{1},{2},{3},{}}
{{1,2,3},{2,3},{1},{2},{3},{}}
{{1,2,3},{1,2},{1,3},{1},{2},{3},{}}
{{1,2,3},{1,2},{2,3},{1},{2},{3},{}}
{{1,2,3},{1,3},{2,3},{1},{2},{3},{}}
{{1,2,3},{1,2},{1,3},{2,3},{1},{2},{3},{}}

CROSSREFS The number of all closure operators is given in A102896.
For T_0 closure operators, see A334252.
For strict T_1 closure operators, see A334255, the only difference is a(1).
Cf. A326960, A326961, A326979.
Sequence in context: A224090 A013327 A009349 * A230582 A011186 A078088
Adjacent sequences: A334251 A334252 A334253 * A334255 A334256 A334257

KEYWORD nonn,more

AUTHOR Joshua Moerman, Apr 20 2020

EXTENSIONS a(6) from Dmitry I. Ignatov, Jul 03 2022

STATUS approved

THE LATTICE OF ATOMIC LATTICES

S. MAPES 2009, 2010

5. Structure of $\mathcal{L}(n)$

In this section, we study some basic properties of $\mathcal{L}(n)$. Counting arguments show that $|\mathcal{L}(3)| = 8$ and $|\mathcal{L}(4)| = 545$, and by using a reverse search algorithm on a computer one can see that $|\mathcal{L}(5)| = 702{,}525$ and $|\mathcal{L}(6)| = 66{,}960{,}965{,}307$ (see appendix A in [Map09]). Thus the complexity of $\mathcal{L}(n)$ rapidly increases with n. Still there are nice properties that we can show about $\mathcal{L}(n)$ which give it some extra structure.

THE LATTICE OF ATOMIC LATTICES

S. MAPES 2009, 2010

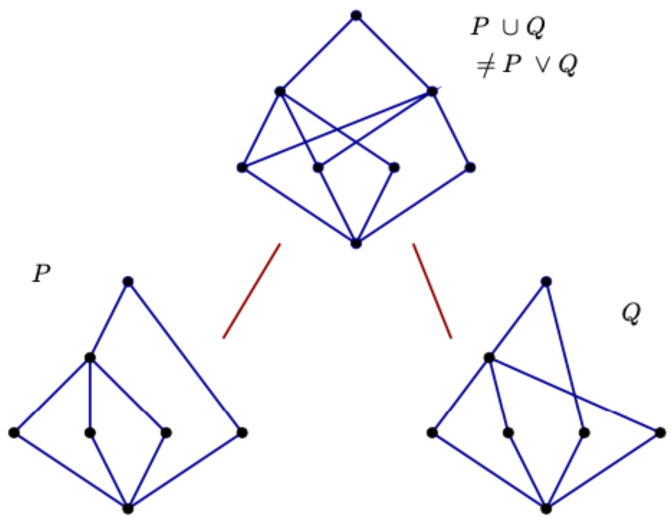


Figure 4.2: $P \cup Q$ is not the join of P and Q

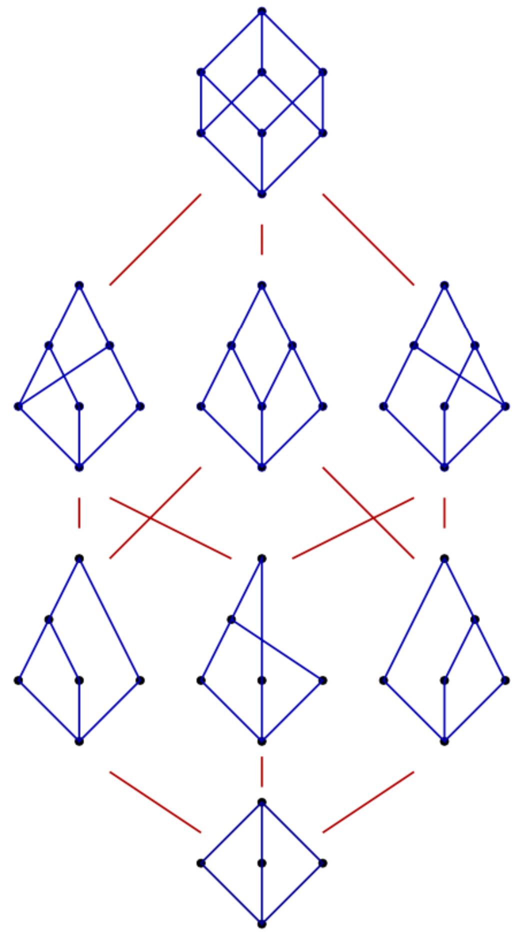
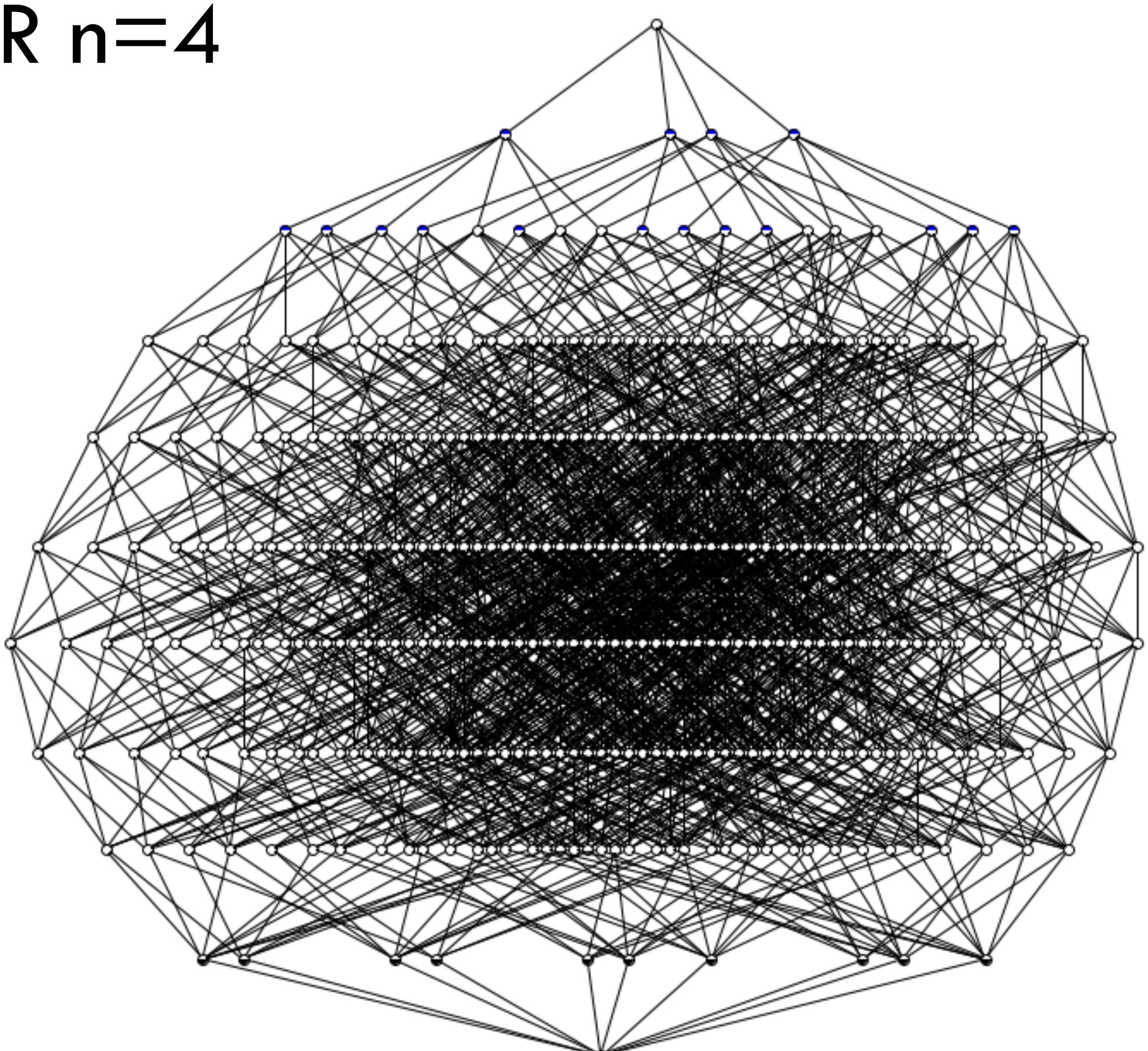


Figure 2.1: $\mathcal{L}(3)$

For $n = 4$, there are 545 elements thus the picture cannot be shown here.

THE LATTICE OF ATOMIC LATTICES FOR n=4



INEQUIVALENT CASE

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0 1 3 6 2 7 13 20 12 21 11 22 10 23 OEIS THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

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Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A235604	Number of equivalence classes of lattices of subsets of the power set 2^[n].
	1, 1, 1, 4, 50, 7443 (list; graph; refs; listen; history; draft edits; text; internal format)
OFFSET	0,4
LINKS	Table of n, a(n) for n=0..5. Donald M. Davis, Enumerating lattices of subsets, arXiv preprint arXiv:1311.6664, 2013
CROSSREFS	Sequence in context: A327229 A231832 A193157 * A221477 A122464 A226375 Adjacent sequences: A235601 A235602 A235603 * A235605 A235606 A235607
KEYWORD	nonn,more,hard
AUTHOR	N. J. A. Sloane, Jan 21 2014
EXTENSIONS	a(5) from Andrew Weimholt, Jan 27 2014
STATUS	approved

INEQUIVALENT CASE
(M. DAVIS, 2014)

ENUMERATING LATTICES OF SUBSETS

DONALD M. DAVIS

Abstract. If $X_1, \ldots, X_k$ are sets such that no one is contained in another, there is an associated lattice on $2^{[k]}$ corresponding to inclusion relations among unions of the sets. Two lattices on $2^{[k]}$ are equivalent if there is a permutation of $[k]$ under which they correspond. We show that for $k = 1, 2, 3,$ and 4, there are 1, 1, 4, and 50 equivalence classes of lattices on $2^{[k]}$ obtained from sets in this way. We cannot find a reference to previous work on this enumeration problem in the literature, and so wish to introduce it for subsequent investigation. We explain how the problem arose from algebraic topology.

INEQUIVALENT CASE
(M. DAVIS, 2014)

Let $[k] = \{1, \ldots, k\}$, and $2^{[k]}$ its power set. If $\mathbb{M} = \{X_1, \ldots, X_k\}$ is a collection of sets, and $S \subset [k]$, let

$$\mathbb{M}_S := \bigcup_{i \in S} X_i. \tag{1.1}$$

We say that $\mathbb{M}$ is *proper* if it is never the case that $X_i \subset X_j$ for $i \neq j$. Any $\mathbb{M}$ defines a lattice $L(\mathbb{M})$ on $2^{[k]}$ by $S \leq T$ if $\mathbb{M}_S \subset \mathbb{M}_T$. Lattices L and L' on $2^{[k]}$ are said to be equivalent if there is a permutation σ of $[k]$ under which the induced permutation of $2^{[k]}$ preserves the lattice relations; i.e., $\sigma(S) \leq' \sigma(T)$ iff $S \leq T$. We wish to enumerate the equivalence classes of all possible $L(\mathbb{M})$'s for proper $\mathbb{M}$'s of size k.

CLOSURE SYSTEMS AND OPERATORS
(GANTER & WILLE, 1999; IGNATOV 2022)

A *closure system* on a set $[n]$ is a set of its subsets which contain $[n]$ and is closed under intersection. That is $\mathcal{M} \subseteq 2^{[n]}$ is a closure system if $[n] \in \mathcal{M}$ and

$$\mathcal{X} \subseteq \mathcal{M} \Rightarrow \bigcap \mathcal{X} \in \mathcal{M}.$$

If a closure system $\mathcal{M}$ contains emptyset, then $\mathcal{M}$ is *strict*.

A *closure operator* φ on $[n]$ is a map assigning a closure $\varphi X \subseteq [n]$ to each subset $X \subseteq [n]$ under the following conditions:

1. $X \subseteq Y \Rightarrow \varphi X \subseteq \varphi Y$ (monotony)
2. $X \subseteq \varphi X$ (extensity)
3. $\varphi\varphi X = \varphi X$ (idempotency)

T1 separation axiom for a closure system $\mathcal{M}$ over $[n]$ states that every single element set $\{i\} \in [n]$ is in $\mathcal{M}$, or, equivalently, is closed, i.e. $\varphi\{i\} = \{i\}$ [17].

Every closure system $\mathcal{M} \subseteq 2^{[n]}$ defines a closure operator as follows:

$$\varphi_{\mathcal{M}} X := \bigcap \{A \in \mathcal{M} \mid X \subseteq A\}.$$

While the set of closures of a closure operator φ is always a closure system $\mathcal{M}_{\varphi}$.

ESTABLISHING CRYPTOMORPHISMS
(IGNATOV 2022)

Theorem 3. *Let $\mathcal{M} \subseteq 2^{[n]}$ be a strict closure system with T1 separation axiom fulfilled, then $(\mathcal{M}, \subseteq)$ is an atomic lattice with $\bigwedge \mathcal{X} = \bigcap \mathcal{X}$ and $\bigvee \mathcal{X} = \varphi_{\mathcal{M}} \bigcup \mathcal{X}$ for all $\mathcal{X} \subseteq \mathcal{M}$. Conversely, every atomic lattice is isomorphic to the lattice of all closures of a strict closure system with T1 separation axiom fulfilled.*

Proposition 4. *Every closure system $\mathcal{M} \subseteq 2^{[n]}$ with T1 separation axiom fulfilled is strict for $n \neq 1$.*

Proof. For $n = 0$ the proposition holds trivially. For $n = 1$ the system $\{\{1\}\}$ is not strict. For $n \geq 2$ any pair $i, j \in [n]$ implies $\{i\} \cap \{j\} = \emptyset$; hence $\emptyset \in \mathcal{M}$. □

ESTABLISHING CRYPTOMORPHISMS
(IGNATOV 2022)

To deal with Davis' lattice, which in fact combines two isomorphic lattices, let us reformulate the original definition.

Let $U = \bigcup_{i\in[n]} X_i$ and R be a binary relation on $[n] \times U$ with iRu if $u \in X_i$.

Consider two operators, $(\cdot)^{\cup} : 2^{[n]} \to 2^U$ and $(\cdot)^{\subseteq} : 2^U \to 2^{[n]}$ that are defined as follows for any $A \subseteq [n]$ and $B \subseteq U$:

$$A^{\cup} := \{u \mid iRu \text{ for some } i \in A\}$$

(the union of all X_i with $i \in A$, i.e., $\mathbb{M}_A$ in Davis' notation)

$$B^{\subseteq} := \{i \mid iRu \text{ implies } u \in B\}$$

(all indices i such that $X_i \subseteq B$).

These two operators $((\cdot)^{\cup}, (\cdot)^{\subseteq})$ forms the so-called *axialities* (cf. Birkhoff's *polarities* [3]), i.e., Galois adjunction [11, 12] between powersets of $[n]$ and U. Note that Galois adjunctions between ordered sets are also known as isotone Galois connections [21].

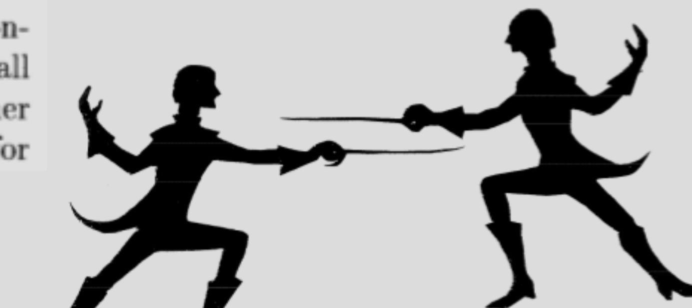
ADJUNCTIONS AND GALOIS CONNECTIONS: ORIGINS, HISTORY AND DEVELOPMENT
(M. ERNE, 2004)

A fundamental fact concerning Galois connections, pointed out in the early sources by Birkhoff [3], Everett [54] and Ore [38], is that all *polarities* in the sense of Birkhoff, that is, all Galois connections between power sets, may be constructed in a unique way from relations between the underlying sets. The partners of the induced Galois connection associate with any subset of the one set the collection of all elements of the other that are in relation to each element of the former subset. We shall focus on that topic in Sections 3.2 and 3.3, but for

1 The Idea of Adjunctions and Galois Connections

Mais je n'ai pas le temps, et mes idées ne sont pas encore bien développées sur ce terrain, qui est immense.

Evariste Galois, in the letter to his friend Auguste Chevalier, written in 1832, on the night before the duel



A Galois Adjunction

A Galois Connection
(Duel Adjunction)

ESTABLISHING CRYPTOMORPHISMS
(IGNATOV, 2022)

Theorem 10. $L(\mathbb{M}) = (\mathcal{M}_{\subseteq\cup}, \subseteq)$ *is atomic lattice. Conversely, every atomic lattice is isomorphic to some* $(\mathcal{M}_{\subseteq\cup}, \subseteq)$.

ESTABLISHING CRYPTOMORPHISMS
(IGNATOV, 2022)

Theorem 10. $L(\mathbb{M}) = (\mathcal{M}_{\subseteq\cup}, \subseteq)$ *is atomic lattice. Conversely, every atomic lattice is isomorphic to some* $(\mathcal{M}_{\subseteq\cup}, \subseteq)$.

EXAMPLES
(IGNATOV, 2022)

Let us consider several binary relations and the lattices of their lower concepts. In Fig. 3.2, one can see three 3×3 exemplary binary relations often used in Formal Concept Analysis for data scaling [14].

	a	b	c
1	×	×	×
2	×	×	
3	×		

	a	b	c
1	×		
2		×	
3			×

	a	b	c
1		×	×
2	×		×
3	×	×	

Figure 1: Example relations for order, nominal, and contranominal scales.

EXAMPLES
(IGNATOV, 2022)

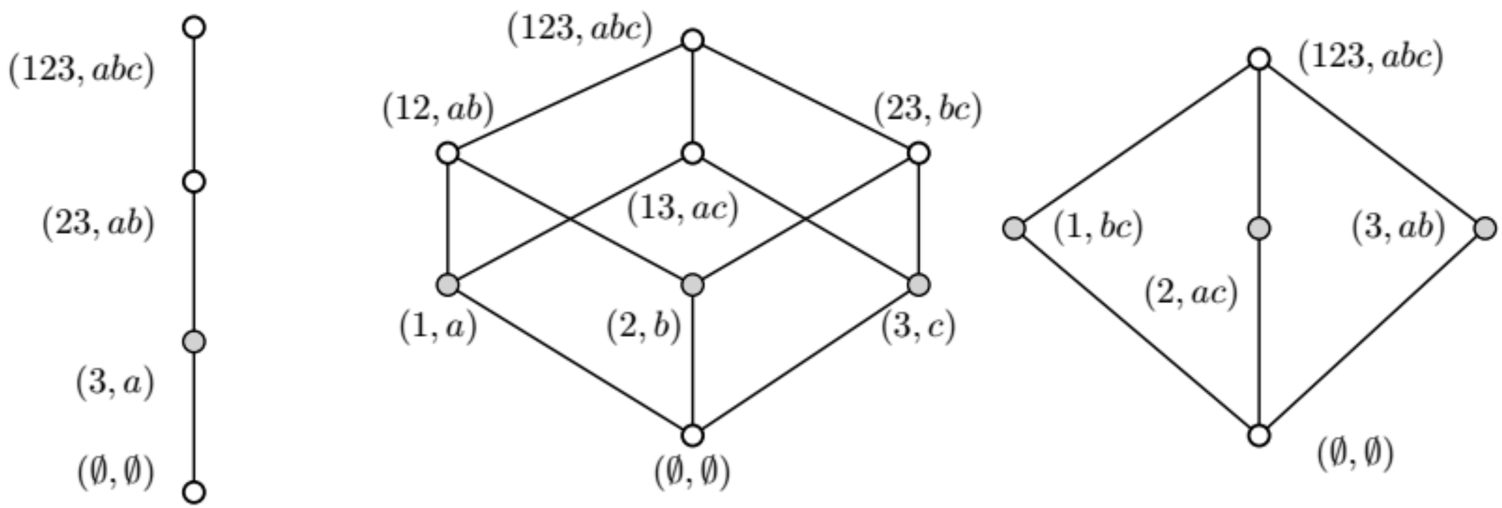


Figure 2: The line diagrams of the lattices of lower concepts for the binary relations in Fig. 3.2, from left to right, respectively.

ALGORITHM
(IGNATOV, 2022)

Since different binary relations $R \subseteq [n] \times U$ with a fixed n can produce the same Moore families on $[n]$ (e.g., by removing a full column $[n] \times \{u\}$ in R if $[n] \times \{u\} \subseteq R$), we need to identify valid ways to reduce U and thus R without affecting the resulting Moore family.

Definition 13 (adopted from Ganter and Wille [14]). We call a binary relation $R \subseteq [n] \times U$ *column reduced* if 1) it is clarified, i.e. R does not contain duplicate rows and columns ($\forall i, j \in [n] : \{i\}^{\cup} = \{j\}^{\cup} \Rightarrow i = j$; similarly, for $u, v \in U$) and 2) there is no $u \in U$, which can be obtained by intersection of other columns $X \subseteq U$, i.e. $u \notin X$ and $\bigcap\limits_{x \in X} x^{\subseteq} \neq u^{\subseteq}$.

A *row reduced* binary relation is defined similarly. If R is both row and column reduced, R is called *reduced*.

In practice, we cannot simultaneously eliminate all the rows and the columns that are *reducible*, but this is no problem if we add rows (or columns) in a lectic order and check reducibility.

By Sperner theorem [28] the largest set antichain in $2^{[n]}$ contains $\binom{n}{\lfloor n/2 \rfloor}$ sets. It makes it possible to deduce the exact lower bound for the number of elements in U for Davis' lattice and the associated relation.

Theorem 14. *The smallest size of U in $R \subseteq [n] \times U$ such that the associated closure system is $T1$-separated (or atomic) is the minimal k under which $\binom{k}{\lfloor k/2 \rfloor} \geq n$.*

ALGORITHM FOR LABELED CASE
(IGNATOV, 2022)

The Atomic AddByOne algorithm is inspired by the CloseByOne algorithm proposed by S. O. Kuznetsov [24].

Algorithm 1. Atomic AddByOne
Input: the number of atoms $n \in \mathbb{N}$ $(n > 1)$
Output: the number of Moore families fulfilling T1 separation axiom

1. Generate all combinations $\binom{2^{[n]} \setminus \{\emptyset, [n]\}}{k_{min}}$ in lectic order.
2. Check each combination represented by a tuple $t = (i_1, \dots, i_k)$ whether it is a column reduced binary relation and fulfils T1 axiom. If yes, store 1 in $cnt[t]$.
3. Extend each valid tuple t (the column reduced binary relation) from step 2 by a next integer i_{k+1} after i_k from $\{i_k + 1, \dots, 2^n - 2\}$ and check whether the new tuple $t^* = (i_1, \dots, i_{k+1})$ is a reduced binary relation and fulfils T1 axiom (if T1 was fulfilled for t, then skip T1-check). If yes, increment $cnt[t]$ and repeat step 3 with t^* recursively.
4. Return the sum of all cnt-s.

Step 1 excludes combinations with emptyset since $\emptyset$ should be present in the resulting system as intersection of atoms by Theorem 3. Since full rows and full columns are reducible, 2^{n-1} is always excluded (every closure system on $[n]$ contains $[n]$ by definition). Note that all subsets of $2^{[n]}$ of size k_{min}, which elements has $\lfloor k_{min}/2 \rfloor$ (or $\lceil k_{min}/2 \rceil$) bits each, forms the antichain of k_{min} elements by Theorem 14 and our previous work on Boolean matrix factorization of contranominal scales [20]. So, Step 1 can be further improved accordingly for n larger than 6 ($k_{min} = 4$).

NON-ISOMORPHIC CASE
(IGNATOV, 2022)

To enumerate inequivalent atomic Moore families, we apply all the permutations $\pi \in \Pi(n)$ on the set $[n]$ to every subset of a concrete Moore family represented by tuple t, i.e. we compute all $\pi(t) = (\pi(i_1), \dots, \pi(i_k))$. We call t *canonic* if it is lectically smallest among all permuted tuples $\pi(t)$. Algorithm 2 counts each canonic representative per an equivalence class w.r.t. $\Pi(n)$.

Algorithm 2. ATOMIC INEQADDBYONE
Input: the number of atoms $n \in \mathbb{N}$ $(n > 1)$
Output: the number of inequivalent Moore families fulfilling T1 separation axiom

The only modification of Algorithm 1 is done at step 3.
3'. We additionally check whether the new tuple t^* is canonic and count only such tuples.

All the implementations are coded in Python, speeded up with Cython extension and multiprocess library, and available on the author's Github[1] along with the results of experiments recorded in Jupyter notebooks.

[1] https://github.com/dimachine/ClosureSeparation

NEW NUMBERS FOR OEIS

n	A334254	A334255	A235604	A355517
0	1	1	1	1
1	2	1	1	2
2	1	1	1	1
3	8	8	4	4
4	545	545	50	50
5	702 525	702 525	7 443	7 443
6	*66 096 965 307*	*66 096 965 307*	*95 239 971*	*95 239 971*

Table 1: Studied sequences with the found extensions in italic.

NEW NUMBERS FOR OEIS

Draft edits for A235604

(Underlined text is an addition; strikethrough text is a ~~deletion~~.)

All edits since published version (omitting small deletions for readability):

NAME	Number of equivalence classes of lattices of subsets of the power set 2^[n].
DATA	1, 1, 1, 4, 50, 7443, 95239971
OFFSET	0,4
COMMENTS	This is also the number of inequivalent atomic lattices on n atoms or inequivalent strict closure systems under T1 separation axiom on n elements. - Dmitry I. Ignatov, Sep 27 2022
LINKS	Donald M. Davis, Enumerating lattices of subsets, arXiv preprint arXiv:1311.6664 [math.CO], 2013. Dmitry I. Ignatov, On the Cryptomorphism between Davis' Subset Lattices, Atomic Lattices, and Closure Systems under T1 Separation Axiom, arXiv:2209.12256 [cs.DM], 2022.
CROSSREFS	The number of inequivalent closure operators on a set of n elements where all singletons are closed is given in A355517. The number of all strict closure operators is given in A102894. For T_1 closure operators, see A334254.
KEYWORD	nonn,more,hard,changed
AUTHOR	N. J. A. Sloane, Jan 21 2014
EXTENSIONS	a(5) from Andrew Weimholt, Jan 27 2014 a(6) from Dmitry I. Ignatov, Sep 27 2022
STATUS	~~approved~~ proposed

EXTRA RESULTS: KLEITMAN'S INTERSECTION FREE FAMILIES
(IN SHORT)

Our by-product is related to the problem of the maximal size union-free set family (the asymptotic was given by D. J. Kleitman [22]), which is dually equivalent to the problem of maximal size intersection-free family or the maximal size of a reduced formal context on n objects as noted by B. Ganter and R. Wille [14]. We have found the value of the latter sequence for $n = 6$ and have provided a concrete lower bound for this value in the case $n = 7$.

EXTRA RESULTS: KLEITMAN'S INTERSECTION FREE FAMILIES
(IN FULL)

Another our contribution is related to the maximal size of the reduced contexts. The first five members for $n = 1, \ldots, 5$ are known in the literature [14]: 1, 2, 4, 7, 13. While we found the sixth term 24 by full enumeration with Algorithms 1 and 2.

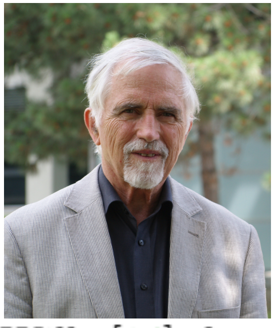
For $n = 6$, one out of ten Moore families of length 24 in its equivalence class is as follows:

$$\{7, 11, 13, 14, 19, 21, 22, 25, 26, 29, 30, 37, 38, 39, 41, 42, 43, 44, 49, 50, 51, 52, 56, 60\}.$$

For the 7th member, we state that it is not less than 41, since by combinatorial (though non-exhaustive) search we found the largest set system of size 41 to form the reduced context of the Moore family: $\{7, 11, 13, 14, 19, 21, 22, 25, 26, 28, 35, 37, 38, 41, 42, 44, 49, 50, 52, 56, 67, 69, 70, 73, 74, 76, 81, 82, 84, 88, 97, 98, 100, 104, 113, 114, 116, 121, 122, 123, 124\}$. In total, 420 Moore families of size 41 are in the found equivalence class.

Note that the last results for $n = 6$ and 7 are also valid for Moore systems without additional constraints.

EXTRA RESULTS: REPRESENTATION OF L_n



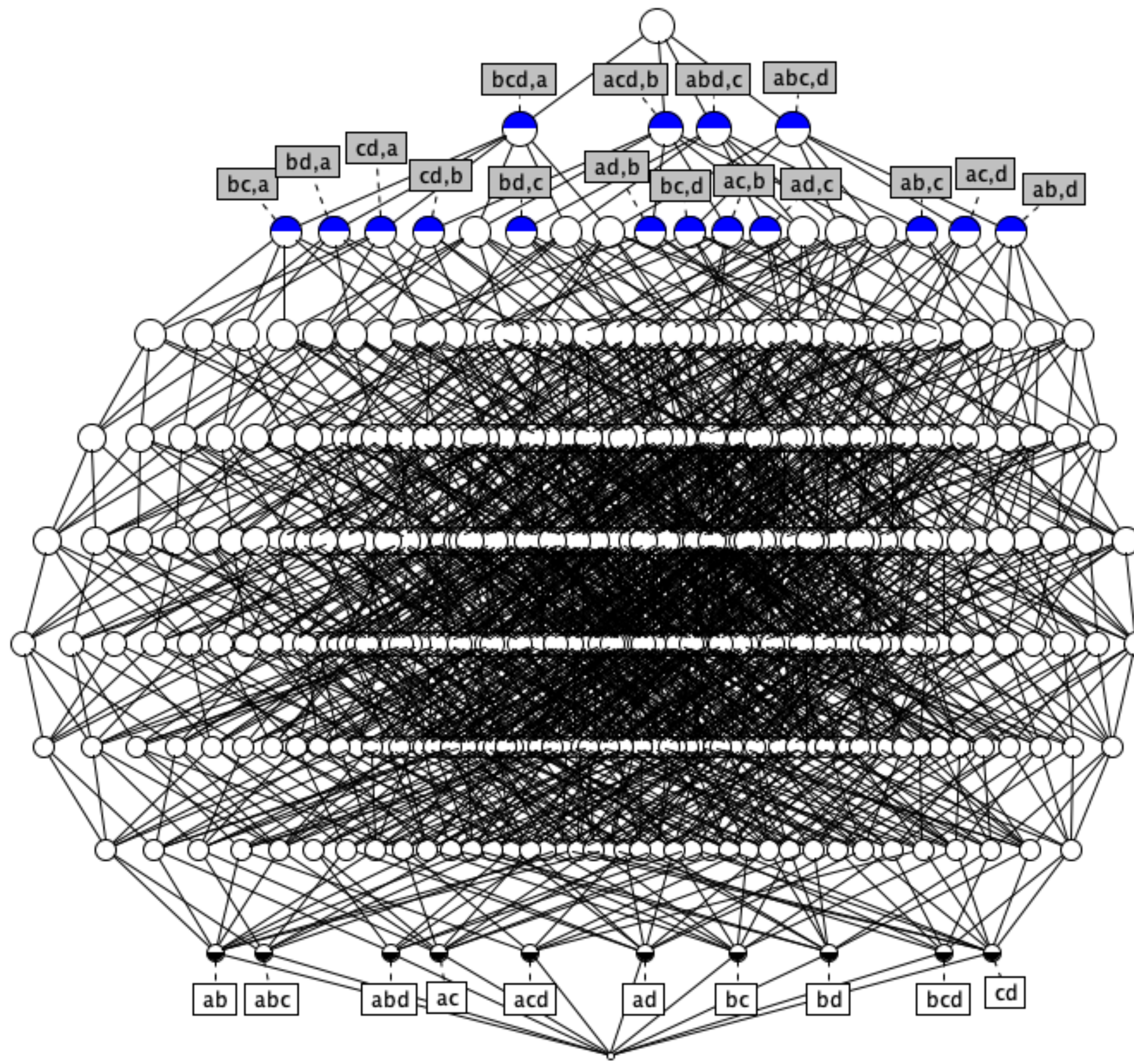
In FCA, Ganter and Wille [14] showed that for any finite lattice $\mathbf{L} := (L, \leq)$, there exists a unique binary relation on join and meet irreducible elements, $J(\mathbf{L})$ and $M(\mathbf{L})$, respectively, such that the lattice formed by all its rows (or columns) closures under intersection is isomorphic to the original lattice; this relation is called a *standard context* and defined as the restriction of $\leq$, i.e. as $\leq \cap J(\mathbf{L}) \times M(\mathbf{L})$. We use $\mathbb{K}(\mathbf{L}) := (J(\mathbf{L}), M(\mathbf{L}), L), \leq)$ to denote the standard context of a lattice $\mathbf{L}$. A similar approach to represent and analyze finite lattices based on a poset of irreducibles is employed by G. Markowsky (e.g., to answer the question from genetics: "What is the smallest number of factors that can be used to represent a given phenotype system?") [8].

EXTRA RESULTS: REPRESENTATION OF L_n
(PHAN, 2006; MAPES 2009; IGNATOV 2022)

Theorem 15. *The number of atoms of the lattice $\mathcal{L}_n$ formed by all atomic closure families on $n > 1$ is equal to $2^n - n - 2$ and each atom has the form $\{\emptyset, \{1\}, \ldots, \{n\}, \sigma, [n]\}$, where $\sigma \subseteq [n]$ and $2 \leq |\sigma| < n$.*

Theorem 16. *Each meet irreducible element in $\mathcal{L}_n$ for $n > 2$ has the form $2^{[n]} \setminus [\sigma, [n] \setminus i]$, where $\sigma \subset [n]$ and $2 \leq |\sigma| < n$ and $i \in [n]$. The number meet irreducible elements for $n \neq 1$ is $n(2^{n-1} - n)$ and 1 for $n = 1$.*

EXTRA RESULTS: LINE DIAGRAM OF L_4
(IGNATOV 2022)



EXTRA RESULTS: STANDARD CONTEXT OF L_n

(IGNATOV 2022)

$\notin$	$[ab, U\setminus c]$	$[ab, U\setminus d]$	$[ac, U\setminus b]$	$[ac, U\setminus d]$	$[bc, U\setminus a]$	$[bc, U\setminus d]$	$[abc, U\setminus d]$	$[ad, U\setminus b]$	$[ad, U\setminus c]$	$[bd, U\setminus a]$	$[bd, U\setminus c]$	$[abd, U\setminus c]$	$[cd, U\setminus a]$	$[cd, U\setminus b]$	$[acd, U\setminus b]$	$[bcd, U\setminus a]$
ab			×	×	×	×	×	×	×	×	×	×	×	×	×	×
ac	×	×			×	×	×	×	×	×	×	×	×	×	×	×
bc	×	×	×	×			×	×	×	×	×	×	×	×	×	×
abc	×		×		×			×	×	×	×	×	×	×	×	×
ad	×	×	×	×	×	×	×			×	×	×	×	×	×	×
bd	×	×	×	×	×	×	×	×	×			×	×	×	×	×
abd		×	×	×	×	×	×	×		×			×	×	×	×
cd	×	×	×	×	×	×	×	×	×	×	×	×			×	×
acd	×	×		×	×	×	×		×	×	×	×	×			×
bcd	×	×	×	×		×	×	×	×		×	×		×	×	

Figure 3: The standard context (10×16) of the lattice of atomic lattices $\mathcal{L}_4$.

$$\left(\bigcup_{k=2}^{n-1} \binom{[n]}{k}, \bigcup_{\substack{i\in[n],\sigma\subseteq[n]\\ |\sigma|\geq 2}} [\sigma, [n]\setminus\{i\}], \notin \right)$$

EXTRA RESULTS: COUNTING FORMAL CONCEPTS
(IGNATOV 2022)

We computed the resulting values of A334254 and A334255 for $n = 3, 4, 5, 6$ with our implementation of parallel NextClosure algorithm (originally proposed by Ganter and Reuter [13]) and thus confirmed the results of Atomic AddByOne. The total computational time is hard to summarise properly per process, but it took about four days for our approach and five days and 17 hours on a laptop with 12 core Intel i-9 processor in parallel mode (seven days and six hours in sequential mode) for NextClosure.

Another interesting approach for enumeration of atomic lattices on n atoms was implemented by S. Mapes in Haskell [25]; even though its running time for $n = 6$ is not reported, it took less than a second for $n = 5$.

EXTRA RESULTS: AN UPPER BOUND FOR $|L_n|$

(IGNATOV 2022)

n	0	1	2	3	4	5	6
A193674, $a(n)$	1	2	7	61	2 480	1 385 552	75 973 751 474
A334254, $b(n)$	1	2	1	8	545	702 525	66 096 965 307
$a(n)/b(n)$	1	1	≈ 0.14	$\approx$0.13	$\approx$0.22	$\approx$0.51	$\approx$0.87

Table 2: Ratio between nth members of sequences A193674 and A334254.

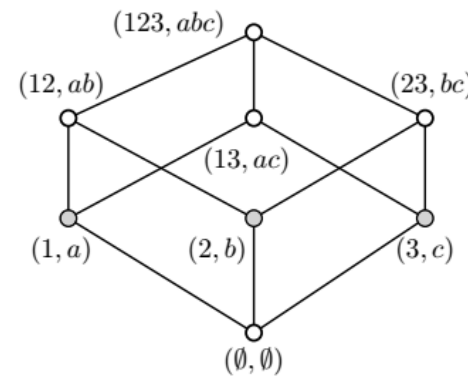
Theorem 18. *Let $\mathcal{L}_n$ and $\mathcal{M}_n$ be the lattices of all atomic Moore families and all Moore families on a set $[n]$ $(n > 1)$, respectively, then*

$$|\mathcal{L}_n| \leq |\mathcal{M}_n| - 2^n - n \ .$$

EXTREMAL LATTICES: BREADTH OF L_n

(IGNATOV 2022)

	a	b	c
1		×	×
2	×		×
3	×	×	



We conclude with visiting another interesting venue, namely, extremal lattice theory, where questions "Why finite lattices described by standard contexts are large?" are studied based on the notion of VC-dimension [1]. As it was shown by Albano and Chornomaz [1], the reason to have a huge number of elements of a lattice is the presence in its standard contexts of the so-called contranominal scales, i.e. induced subcontexts (subrelations) of the form $\mathbb{N}^c(k) := (\{1, \ldots, k\}, \{1, \ldots, k\}, \neq)$ (e.g., the rightmost binary relation in Fig. 1 is $\mathbb{N}^c(3)$).

For example, the closure systems generated by a contranominal scale on n elements taken as a standard context has 2^n closed sets.

The *breadth* of a complete lattice is the number of atoms of the largest Boolean lattice that the lattice contains as a suborder, i.e. the size of the largest contranominal scale subrelation of its standard context (valid for all finite contexts) as noted by Ganter [16].

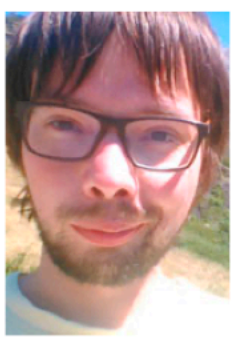
EXTREMAL LATTICES : BREADTH OF L_n
(IGNATOV 2022)

Theorem 19 (Albano and Chornomaz [1]). *Let* $\mathbb{K} := ([n], U, I \subseteq [n] \times U)$ *be an* $\mathbb{N}^c(k)$*-free formal context, then*

$$|\mathbf{L}(\mathbb{K})| \leq \sum_{i=0}^{k-1} \binom{n}{i}.$$

In what follows, we denote the sum from Theorem 19 by $f_{AC}(n, k)$.

Alexandre Albano received his MSc in Computer Science. He has worked as a teaching assistant at the University of São Paulo and at the Dresden University of Technology, where he has recently submitted his PhD thesis. His research interests mainly belong to the fields of Formal Concept Analysis, graph theory, and knowledge representation systems.



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EXTREMAL LATTICES : BREADTH OF L_n
(IGNATOV 2022)

n	3	4	5	6	7
$\|J(\mathcal{L}_n)\|$	3	10	25	56	119
$\|J(\mathcal{M}_n)\|$	7	15	31	63	127
Estimated breadth of $\mathcal{L}_n$	3	5	7	11	?
Estimated breadth of $\mathcal{M}_n$	3	5	7	10	16
Breadth of $\mathcal{L}_n$	3	7	13	≥ 18	≥ 25

Table 3: Estimated breadths of lattices $\mathcal{L}_n$ and $\mathcal{M}_n$ for $n = 3$ up to 6 and 7, respectively.

EXTREMAL LATTICES : BREADTH OF L_n (IGNATOV 2022)

By using this theorem and our knowledge on the number of meet-irreducible element for the lattice of all atomic Moore families and Moore families, respectively, we obtain two series (Table 3) for the estimated breadth of those two lattices for known $|\mathcal{L}_n|$ and $|\mathcal{M}_n|$. Note that $|J(\mathcal{M}_n)| = 2^n - 1$ follows from Definition 17 and Proposition 18 in Caspard and Monjardet [5].

For example, for $n = 6$ we know $|J(\mathcal{L}_6)| = 56$. For $k = 11$ we get $f_{AC}(56, 11) = \sum_{i=0}^{10} \binom{56}{i} = 44872116214 < |\mathcal{L}_6| = 66096965307$, while for $k = 12$ we have $f_{AC}(n,k)(56, 12) = 193774331494 > |\mathcal{L}_6|$. So, $\mathcal{L}_6$ contains a 2048-elements Boolean lattice and the breadth of $\mathcal{L}_6$ is at least 11.

The actual breadth of $\mathcal{L}_3$ is indeed 3 since the standard context of $\mathcal{L}_3$ coincides with $\mathbb{N}^c(3)$, while the actual breadth of $\mathcal{L}_4$ is 7 since there are 80 embedded $\mathbb{N}^c(7)$ and that of $\mathcal{L}_5$ is 13 since there are 10980 embedded $\mathbb{N}^c(13)$ (the last line in Table 3 is found by a combinatorial search).

Thank you!

Questions?