Feynman checkers: Minkowskian lattice quantum field theory

M. Skopenkov[1,2] A. Ustinov[1,3]

[1]National Research University Higher School of Economics

[2]Institute for Information Transmission Problems RAS

[3]Institute of applied mathematics Khabarovsk division
Far Eastern branch of Russian Academy of Sciences

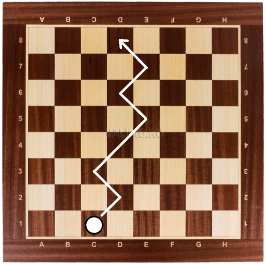
Austrian–Russian Meeting

Overview

1. Feynman's quantum mechanical model
2. Consistency with the continuum theory
3. A new quantum field theory model
4. Consistency with the continuum theory

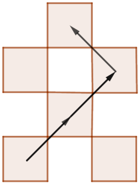
Feynman's quantum mechanical model

Informal illustration



path $\mapsto$ vector

Informal illustration



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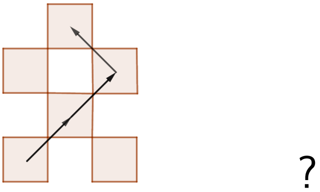
Informal illustration



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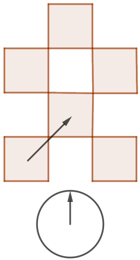

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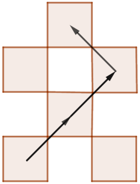

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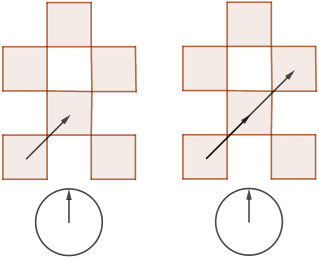
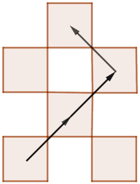
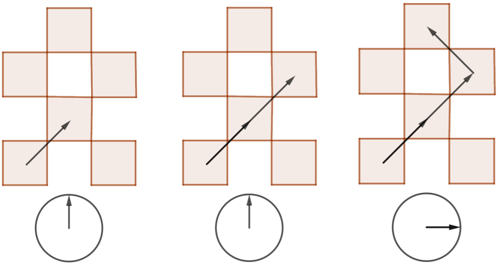




$$\text{path} \quad \mapsto \quad \overset{?}{\text{vector}}$$



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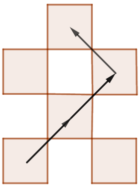

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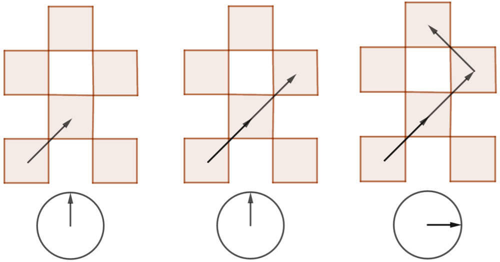
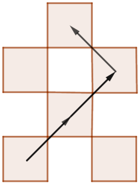


$\leftarrow$ direction

Informal illustration



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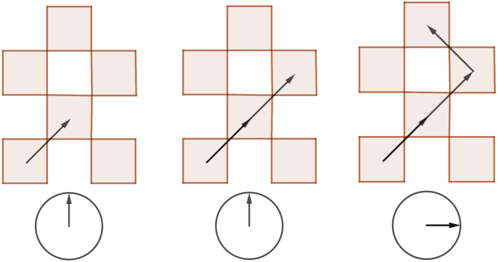

length $= \frac{1}{2^{(t-1)/2}}$,
where $t =$
number of moves

$\leftarrow$ direction

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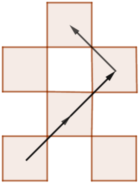

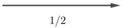

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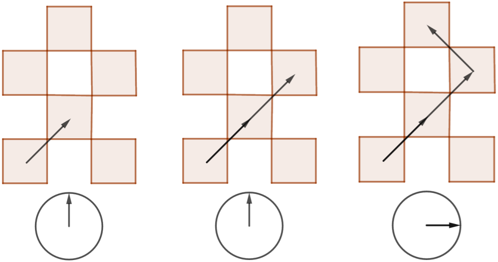

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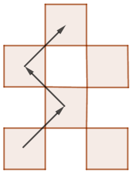

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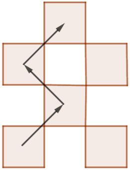

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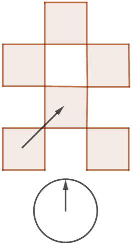

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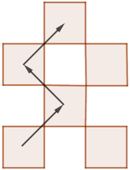

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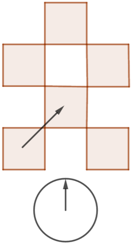

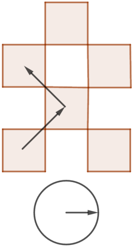

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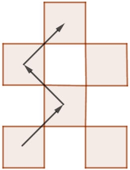

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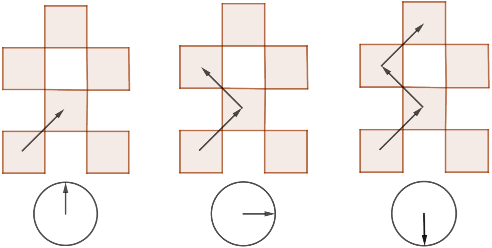

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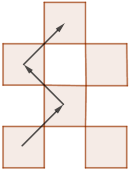

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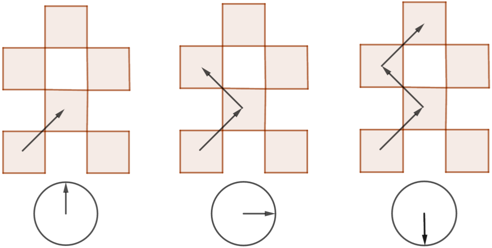

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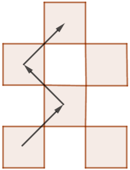
Informal illustration



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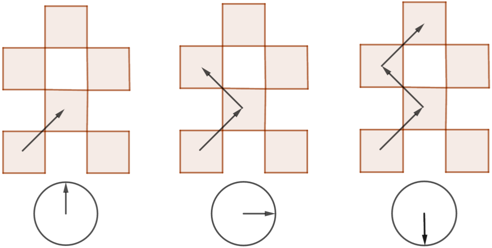

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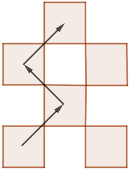

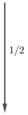

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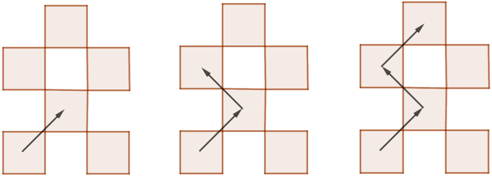

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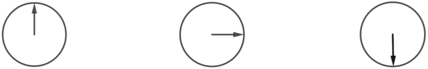

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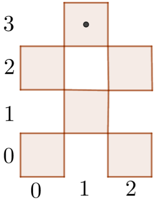
Informal illustration



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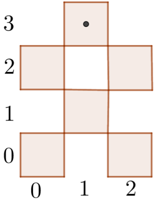
square $\mapsto$ vector

Informal illustration



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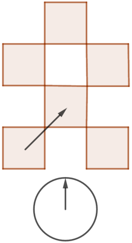
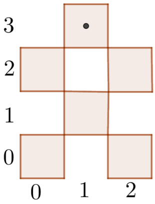
square $\mapsto$ vector



all paths from the origin
to the square starting
with the upwards-right move

Informal illustration



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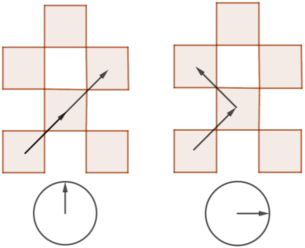
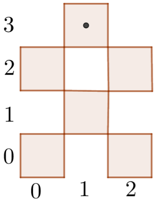
square $\mapsto$ vector





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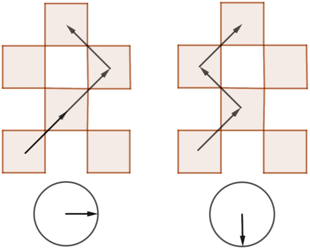
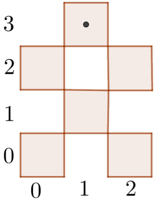
square $\mapsto$ vector



Informal illustration



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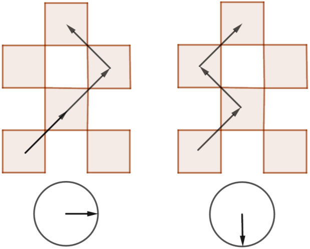
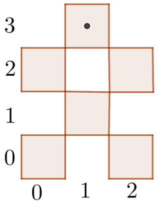
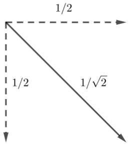


vector of a square =
sum of vectors
of paths

Informal illustration





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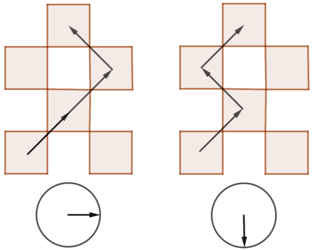

vector of a square $=$
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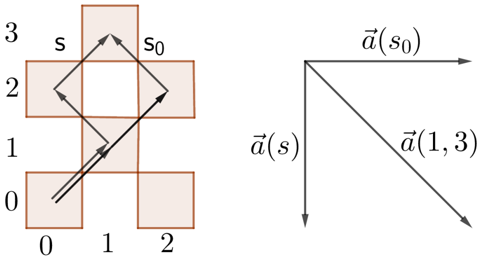
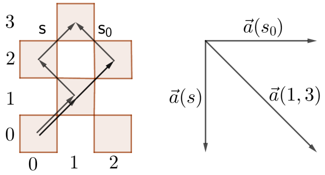


The length square of the resulting vector is the *probability to find an electron in the square* (x, t), if it was emitted from $(0, 0)$:

$$P(1,3) = |a(1,3)|^2 = 1/2.$$

Definition (R.Feynman, 1950s)

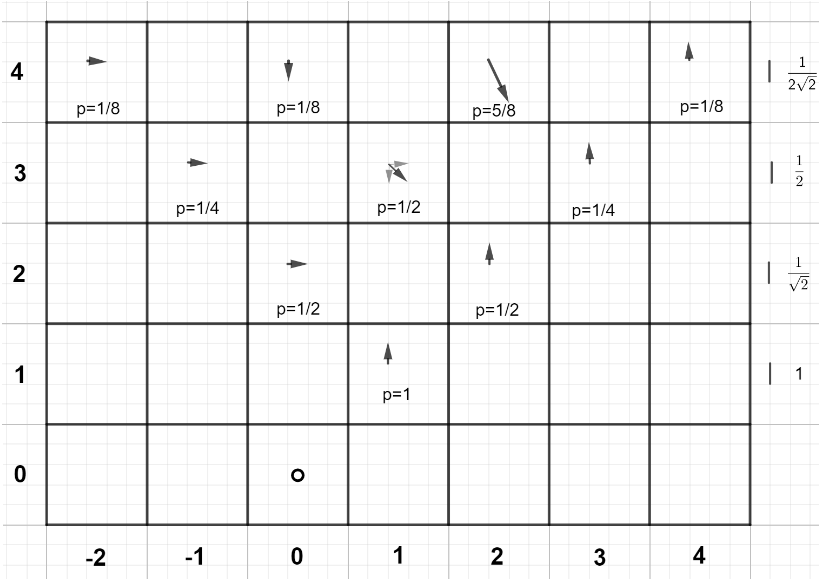


$$a(x,t) := 2^{(1-t)/2}\, i \sum_s (-i)^{\mathrm{turns}(s)}$$

is the sum over all checker paths s from $(0,0)$ to (x,t) with the first step to $(1,1)$, where $\mathrm{turns}(s)$ is the number of turns in s.

$$P(x,t) := |a(x,t)|^2.$$

$a(x,t)$ and $P(x,t)$ for small x,t (V. Skopenkova)

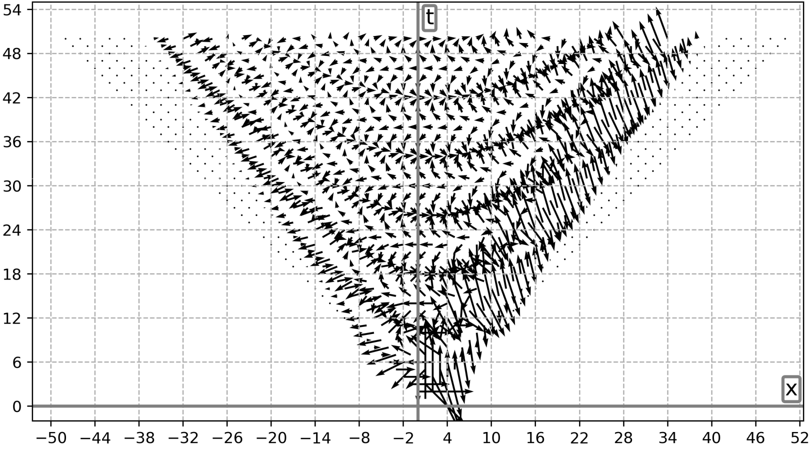


Proposition (folklore)

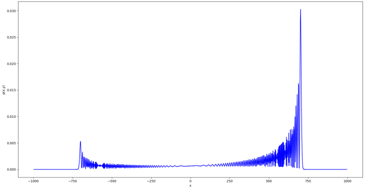
For each integer $t \geq 1$ we have

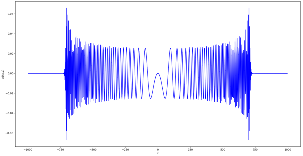
$$\sum_{x \in \mathbb{Z}} P(x, t) = 1.$$

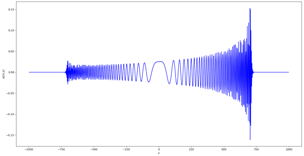
$10a(x,t)$ for $t \leq 50$ (M. Fedorov)



$P(x, 1000)$, $\mathrm{Re}\, a(x, 1000)$, $\mathrm{Im}\, a(x, 1000)$ (A. Daniyarkhodzhaev–F. Kuyanov)

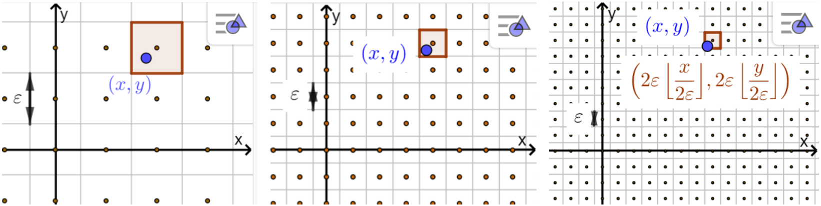




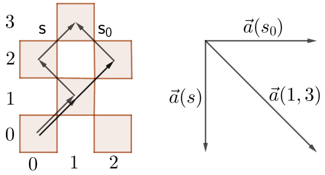


Consistency with the continuum theory

Setup



$$\varepsilon\mathbb{Z}^2 = \{\, (x,t) : x/\varepsilon, t/\varepsilon \in \mathbb{Z} \,\}$$



$$a(x,t) := 2^{(1-t)/2}\, i \sum_{s} (-i)^{\mathrm{turns}(s)}$$

is the sum over all checker paths s from $(0,0)$ to (x,t) with the first step to $(1,1)$, where $\mathrm{turns}(s)$ is the number of turns in s.

$$P(x,t) := |a(x,t)|^2.$$

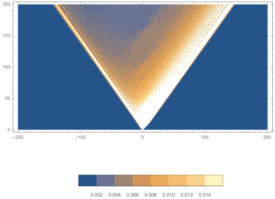
Fix $\varepsilon, m > 0$ (*lattice step* and *particle mass*).
For each $(x, t) \in \varepsilon\mathbb{Z}^2$ set

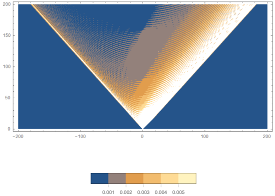
$$a(x,t,m,\varepsilon) := (1+m^2\varepsilon^2)^{\frac{1-t/\varepsilon}{2}}\, i \sum_s (-im\varepsilon)^{\mathrm{turns}(s)}$$

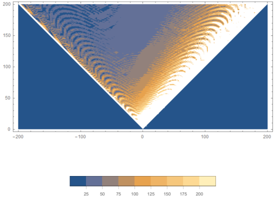
is the sum over all checker paths s from $(0,0)$ to (x,t) with the first step to $(\varepsilon,\varepsilon)$.

$$P(x,t,m,\varepsilon) := |a(x,t,m,\varepsilon)|^2.$$

$P(x,t,1,1)$, $P(x,t,1,0.5)$, and the continuum analogue







Dirac equation

Denote $\begin{aligned} a_1(x,t) &:= \operatorname{Re} a(x,t,m,\varepsilon), \\ a_2(x,t) &:= \operatorname{Im} a(x,t,m,\varepsilon). \end{aligned}$

Proposition (folklore)

For each $x, t \in \varepsilon\mathbb{Z}$, $t > 0$, we have

$$a_1(x,t) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_1(x+\varepsilon, t-\varepsilon) + m\varepsilon\, a_2(x+\varepsilon, t-\varepsilon)),$$
$$a_2(x,t) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_2(x-\varepsilon, t-\varepsilon) - m\varepsilon\, a_1(x-\varepsilon, t-\varepsilon)).$$

Cf. $(1+1)$-dimensional Dirac equation

$$\begin{pmatrix} m & \partial/\partial x - \partial/\partial t \\ \partial/\partial x + \partial/\partial t & m \end{pmatrix} \begin{pmatrix} a_2(x,t) \\ a_1(x,t) \end{pmatrix} = 0$$

A new quantum field theory model

New feature: electron-positron pairs can be created and annihilated during the motion

Implementation: *second quantization*, which we do not discuss as we only need the

Result: the propagator acquires an imaginary part nonvanishing even for $x > t$.

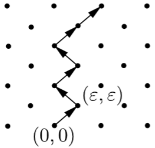
Start with *lattice Dirac equation*

$$a_1(x,t) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_1(x+\varepsilon, t-\varepsilon) + m\varepsilon\, a_2(x+\varepsilon, t-\varepsilon)),$$

$$a_2(x,t) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_2(x-\varepsilon, t-\varepsilon) - m\varepsilon\, a_1(x-\varepsilon, t-\varepsilon)).$$

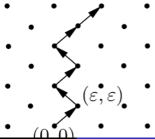
and modify it in 3 steps:

1. a_1 and a_2 are extended to the *dual* lattice, $\pm\varepsilon$ are replaced by $\pm\varepsilon/2$, a term vanishing outside $(0,0)$ is added;



Modification:

1. a_1 and a_2 are extended to the *dual* lattice, $\pm\varepsilon$ are replaced by $\pm\varepsilon/2$, a term vanishing outside $(0,0)$ is added;
2. particle mass acquires small imaginary part which we eventually tend to zero;
3. on the dual lattice, the mass is replaced by its imaginary part.



Fix $\varepsilon, m > 0, 0 < \delta < 1$ called *lattice step, particle mass*, and *small imaginary mass*. Define a pair of complex-valued functions $A_k(x,t) = A_k(x,t,m,\varepsilon,\delta)$, $k \in \{1,2\}$, on $\{\,(x,t) \in \mathbb{R}^2 : 2x/\varepsilon, 2t/\varepsilon, (x+t)/\varepsilon \in \mathbb{Z}\,\}$ by the following 3 conditions:

Definition of the new model

1. for each (x,t) with $2x/\varepsilon$ and $2t/\varepsilon$ even,

$$A_1(x,t) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}\left(A_1\left(x+\frac{\varepsilon}{2}, t-\frac{\varepsilon}{2}\right) + m\varepsilon\, A_2\left(x+\frac{\varepsilon}{2}, t-\frac{\varepsilon}{2}\right)\right),$$

$$A_2(x,t) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}\left(A_2\left(x-\frac{\varepsilon}{2}, t-\frac{\varepsilon}{2}\right) - m\varepsilon\, A_1\left(x-\frac{\varepsilon}{2}, t-\frac{\varepsilon}{2}\right)\right) + \\ + 2\delta_{x0}\delta_{t0};$$

2. for each (x,t) with $2x/\varepsilon$ and $2t/\varepsilon$ odd,

$$A_1(x,t) = \frac{1}{\sqrt{1-\delta^2}}\left(A_1\left(x+\frac{\varepsilon}{2}, t-\frac{\varepsilon}{2}\right) - i\delta\, A_2\left(x+\frac{\varepsilon}{2}, t-\frac{\varepsilon}{2}\right)\right),$$

$$A_2(x,t) = \frac{1}{\sqrt{1-\delta^2}}\left(A_2\left(x-\frac{\varepsilon}{2}, t-\frac{\varepsilon}{2}\right) + i\delta\, A_1\left(x-\frac{\varepsilon}{2}, t-\frac{\varepsilon}{2}\right)\right);$$

3. $\sum_{(x,t)\in\varepsilon\mathbb{Z}^2}\left(|A_1(x,t)|^2 + |A_2(x,t)|^2\right) < \infty.$

Set $\widetilde{A}_k(x,t,m,\varepsilon) := \lim_{\delta\searrow 0} A_k(x,t,m,\varepsilon,\delta)$.

Theorem (S, U, 2022)

Both $A_k(x,t,m,\varepsilon,\delta)$ *and* $\widetilde{A}_k(x,t,m,\varepsilon)$ *are well-defined. For* $(x+t)/\varepsilon+k$ *even,* $\widetilde{A}_k(x,t,m,\varepsilon)$ *is real and given by*

$$\widetilde{A}_1(x,t,m,\varepsilon) = a_1(x,|t|+\varepsilon), \qquad 2\nmid\frac{x+t}{\varepsilon},$$
$$\widetilde{A}_2(x,t,m,\varepsilon) = \pm a_2(\pm x+\varepsilon,|t|+\varepsilon), \quad 2\mid\frac{x+t}{\varepsilon},$$

where the minus signs are taken when $t<0$*. For* $(x+t)/\varepsilon+k$ *odd, it is purely imaginary.*

Physical interpretation

One interprets

$$\frac{1}{2}\left|\widetilde{A}_1(x,t,m,\varepsilon)\right|^2 + \frac{1}{2}\left|\widetilde{A}_2(x,t,m,\varepsilon)\right|^2$$

as the *expected charge* in the interval of length ε around the point x at the time $t > 0$, if an electron of mass m was emitted from the origin at the time 0 (or a positron is absorbed there).

This value cannot be interpreted as probability anymore.

Exact solution

Proposition (S, U, 2022)

For each $x, t \in \varepsilon\mathbb{Z}$ *we have*

$$\widetilde{A}_1(x,t,m,\varepsilon) := \pm \frac{im\varepsilon^2}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{e^{ipx - i\omega_p t}\, dp}{\sqrt{m^2\varepsilon^2 + \sin^2(p\varepsilon)}},$$

$$\widetilde{A}_2(x,t,m,\varepsilon) := \pm \frac{\varepsilon}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \left(1 + \frac{\sin(p\varepsilon)}{\sqrt{m^2\varepsilon^2 + \sin^2(p\varepsilon)}}\right) e^{ipx - i\omega_p t}\, dp,$$

where the minus sign in the expression for $\widetilde{A}_k$ *is taken when* $t < 0$ *and* $(x+t)/\varepsilon + k$ *is even, and* $\omega_p := \frac{1}{\varepsilon} \arccos\left(\frac{\cos p\varepsilon}{\sqrt{1+m^2\varepsilon^2}}\right)$.

$$G^F(x,t) = \int_{-\infty}^{+\infty} \begin{pmatrix} \frac{im}{\sqrt{m^2+p^2}} & 1 + \frac{p}{\sqrt{m^2+p^2}} \\ \frac{p}{\sqrt{m^2+p^2}} - 1 & \frac{im}{\sqrt{m^2+p^2}} \end{pmatrix} \frac{e^{ipx - i\sqrt{m^2+p^2}\,t}\, dp}{4\pi}, \quad t > 0$$

Table of $\widetilde{A}_1(x, t, 1, 1)$

$t \backslash x$	-1	0	1	2	3
2	$\frac{1}{2}$		$\frac{1}{2}$		0
1		$\frac{1}{\sqrt{2}}$		0	
0	0		0		0
-1		$\frac{1}{\sqrt{2}}$		0	

Table of $\widetilde{A}_1(x,t,1,1)$

2	$\frac{1}{2}$		$\frac{1}{2}$		0
1		$\frac{1}{\sqrt{2}}$		0	
0	0	iG	0		0
-1		$\frac{1}{\sqrt{2}}$		0	
$t \backslash x$	-1	0	1	2	3

$G := \Gamma(\frac{1}{4})^2/(2\pi)^{3/2}$ is the *Gauss constant*

Table of $\widetilde{A}_1(x,t,1,1)$

2	$\frac{1}{2}$	$-iL'$	$\frac{1}{2}$		0
1		$\frac{1}{\sqrt{2}}$		0	
0	0	iG	0		0
-1		$\frac{1}{\sqrt{2}}$		0	
$t \backslash x$	-1	0	1	2	3

$G := \Gamma(\frac{1}{4})^2/(2\pi)^{3/2}$ is the *Gauss constant*

$L' := 1/\pi G$ is the *inverse lemniscate constant*

Table of $\widetilde{A}_1(x, t, 1, 1)$

2	$\frac{1}{2}$	$-iL'$	$\frac{1}{2}$	$i\frac{2G-3L'}{3}$	0
1	$i\frac{G-L'}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$i\frac{G-L'}{\sqrt{2}}$	0	$i\frac{7G-15L'}{3\sqrt{2}}$
0	0	iG	0	$i(G-2L')$	0
-1	$i\frac{G-L'}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$i\frac{G-L'}{\sqrt{2}}$	0	$i\frac{7G-15L'}{3\sqrt{2}}$
$t \diagup x$	-1	0	1	2	3

$G := \Gamma(\frac{1}{4})^2/(2\pi)^{3/2}$ is the *Gauss constant*
$L' := 1/\pi G$ is the *inverse lemniscate constant*

“Explicit formula”

$$\mathrm{Im}\widetilde{A}_1(x,t,m,\varepsilon) = \left(1+m^2\varepsilon^2\right)^{-\frac{t}{2\varepsilon}} \left(-m^2\varepsilon^2\right)^{\frac{t-|x|}{2\varepsilon}} \binom{\frac{t+|x|}{2\varepsilon}-\frac{1}{2}}{|x|/\varepsilon}$$

$$\cdot\, {}_2F_1\left(\frac{1}{2}+\frac{|x|-t}{2\varepsilon}, \frac{1}{2}+\frac{|x|-t}{2\varepsilon}; 1+\frac{|x|}{\varepsilon}; -\frac{1}{m^2\varepsilon^2}\right),$$

$$\mathrm{Im}\widetilde{A}_2(x,t,m,\varepsilon) = \left(1+m^2\varepsilon^2\right)^{-\frac{t}{2\varepsilon}} (m\varepsilon)^{\frac{t-|x|}{\varepsilon}} (-1)^{\frac{t-|x|}{2\varepsilon}+\frac{1}{2}} \binom{\frac{t+|x|}{2\varepsilon}-1+\theta(x)}{|x|/\varepsilon}$$

$$\cdot\, {}_2F_1\left(\frac{|x|-t}{2\varepsilon}, 1+\frac{|x|-t}{2\varepsilon}; 1+\frac{|x|}{\varepsilon}; -\frac{1}{m^2\varepsilon^2}\right),$$

for $(x+t)/\varepsilon$ even and odd respectively, where

$$\theta(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0; \end{cases}$$

and ${}_2F_1(p,q;r;z)$ is the principal branch of the Gauss hypergeometric function.

Consistency with the continuum theory

Main theorem

Theorem (S, U, 2022)

For each $m, \varepsilon > 0$ and $(x,t) \in \varepsilon\mathbb{Z}^2$ such that $|x| \neq |t|$ we have

$$\widetilde{A}_1(x,t,m,\varepsilon) = \begin{cases} m\varepsilon\left(J_0(ms) + O(\varepsilon\Delta)\right), & \text{for } |x| < |t|, \frac{x+t}{\varepsilon} \text{ odd;} \\ -im\varepsilon\left(Y_0(ms) + O(\varepsilon\Delta)\right), & \text{for } |x| < |t|, \frac{x+t}{\varepsilon} \text{ even;} \\ 0, & \text{for } |x| > |t|, \frac{x+t}{\varepsilon} \text{ odd;} \\ 2im\varepsilon\left(K_0(ms) + O(\varepsilon\Delta)\right)/\pi, & \text{for } |x| > |t|, \frac{x+t}{\varepsilon} \text{ even;} \end{cases}$$

$$\widetilde{A}_2(x,t,m,\varepsilon) = \begin{cases} -m\varepsilon(t+x)\left(J_1(ms) + O(\varepsilon\Delta)\right)/s, & \text{for } |x| < |t|, \frac{x+t}{\varepsilon} \text{ even;} \\ im\varepsilon(t+x)\left(Y_1(ms) + O(\varepsilon\Delta)\right)/s, & \text{for } |x| < |t|, \frac{x+t}{\varepsilon} \text{ odd;} \\ 0, & \text{for } |x| > |t|, \frac{x+t}{\varepsilon} \text{ even;} \\ 2im\varepsilon(t+x)\left(K_1(ms) + O(\varepsilon\Delta)\right)/\pi s, & \text{for } |x| > |t|, \frac{x+t}{\varepsilon} \text{ odd.} \end{cases}$$

where $s := \sqrt{|t^2 - x^2|}$ *and* $\Delta := \dfrac{1}{\big||x| - |t|\big|} + m^2(|x| + |t|).$

Spin-1/2 Feynman propagator $G^F(x,t) =$

$$= \begin{cases} \frac{m}{4} \begin{pmatrix} J_0(ms) - iY_0(ms) & -\frac{t+x}{s}\left(J_1(ms) - iY_1(ms)\right) \\ \frac{t-x}{s}\left(J_1(ms) - iY_1(ms)\right) & J_0(ms) - iY_0(ms) \end{pmatrix}, & |x| < |t|; \\ \frac{im}{2\pi} \begin{pmatrix} K_0(ms) & \frac{t+x}{s} K_1(ms) \\ \frac{x-t}{s} K_1(ms) & K_0(ms) \end{pmatrix}, & |x| > |t|; \end{cases}$$

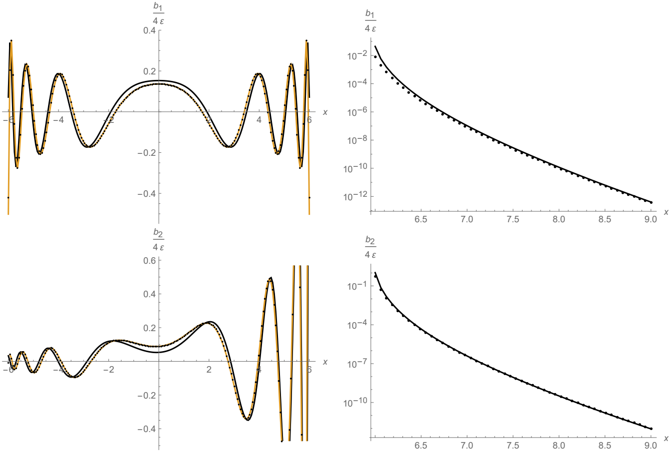
$+$ a generalized function supported on $\{t = \pm x\}$.

Here:

- $J_n(z) :=$ Bessel functions of the 1st kind;
- $Y_n(z) :=$ Bessel functions of the 2nd kind;
- $K_n(z) :=$ modified Bessel functions;
- $s := \sqrt{|t^2 - x^2|}$.

Consistency with continuum theory

Plots of the discrete propagator $b_k(x) := \mathrm{Im}\widetilde{A}_k(x, 6, 4, 0.03)$ and the continuum one $G^F_{1k}(x, 6)$:



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