On statistical properties of 3D Voronoi-Minkowski continued fractions

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"Diophantine Analysis, Dynamics and Related Topics"
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Georgy Voronoï (1868–1908) and Hermann Minkowski (1864-1909) looked pretty similar and shared nearly parallel biographies (including their untimely death);
they met once at the ICM in Heidelberg 1904. They founded Geometry of Numbers – a new branch of mathematics, around 1895.

MINKOWSKI H. Generalisation de la theorie des fractions continues. — *Ann. de l'Ecole Norm.*, 1896, 13, 41–60.

VORONOÏ G. F. *On a Generalization of the Algorithm of Continued Fractions (Doctoral Dissertation)*. — Warsaw, 1896. (195 pp. in reprinted edition)

Two translations of Voronoï's thesis are available:

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DELONE, B. N. FADDEEV D. K. Theory of irrationalities of third degree, — *Travaux Inst. Math. Stekloff*, 11, Acad. Sci. USSR, Moscow-Leningrad, 1940.

VORONOÏ G. F. *On a Generalization of the Algorithm of Continued Fractions (Doctoral Dissertation)* — unofficial translation by Emma Lehmer (exists as pdf-document).

"The present work was completely finished and the printing begun when there was received in Warsaw No. 2 13th Vol. of *Annales Scientifique de l'École Normale Supérieure*. In this no. is the article by H. Minkowski 'Généralisation de la théorie des fractions continues'. . . "

G. F. Voronoï, Warsaw 24th May 1896."

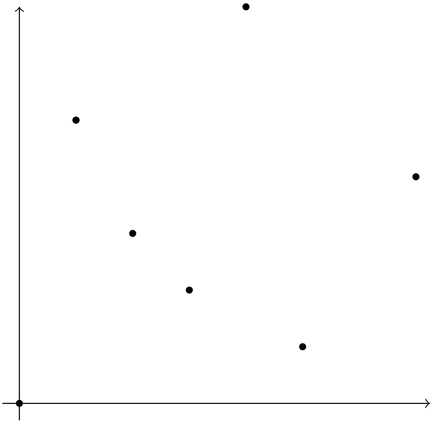
Previous 3D generalizations of continued fraction algorithm were considered by

- Euler (???),
- Jacobi (1868),
- Hermite (1845),
- Poincaré (1885),
- Hurwitz (1894),
- Klein (1895).

Let S be a subset of $\mathbb{R}^2_{\geqslant 0}$. Consider the boundary of the set

$$S \oplus \mathbb{R}^2_{\geqslant 0} = \{s + r \mid s \in S, r \in \mathbb{R}^2_{\geqslant 0}\}.$$

In other words, this broken line is the boundary of the union of copies of the positive quadrant shifted by vertices of the set S.



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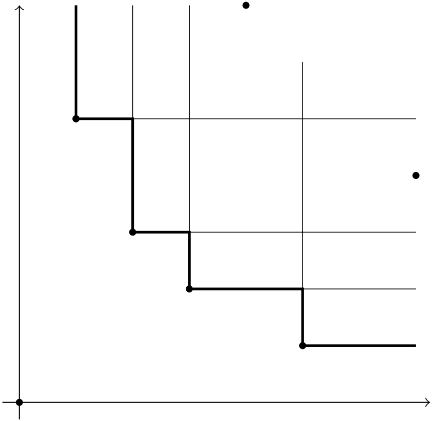

Let S be a subset of $\mathbb{R}^3_{\geqslant 0}$. The *Voronoi-Minkowski polyhedron* for S is the boundary of the set

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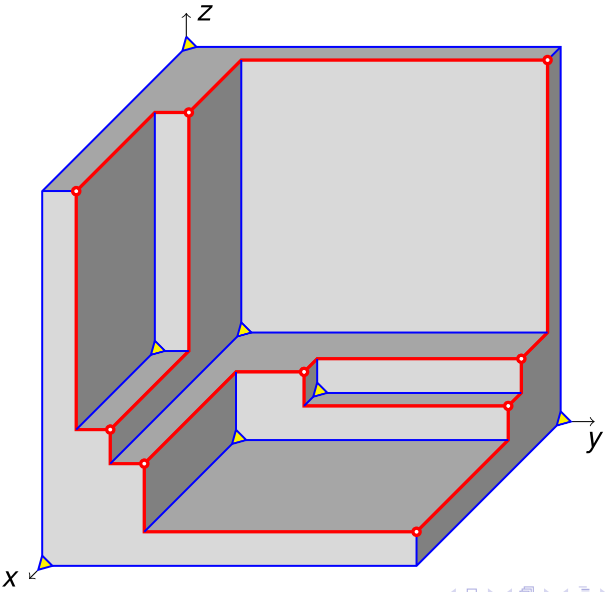
We assume that

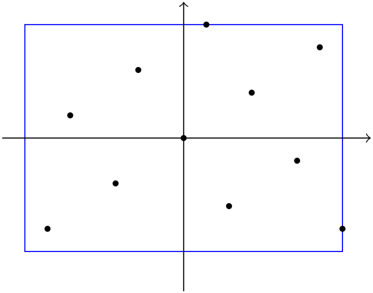
(1) S has no accumulation points;

(2) S is in general position: each plane parallel to a coordinate plane contains at most one point of S.

Voronoi-Minkowski complex



For a nonempty finite point set $T \subset \mathbb{R}^s$ $\mathrm{Box}(T)$ is the least possible parallelepiped circumscribed about T.



More formally: if

$$|T|_i = \max\{|x_i| : \ x = (x_1, \ldots, x_s) \in T\} \quad (i = 1, \ldots, s),$$

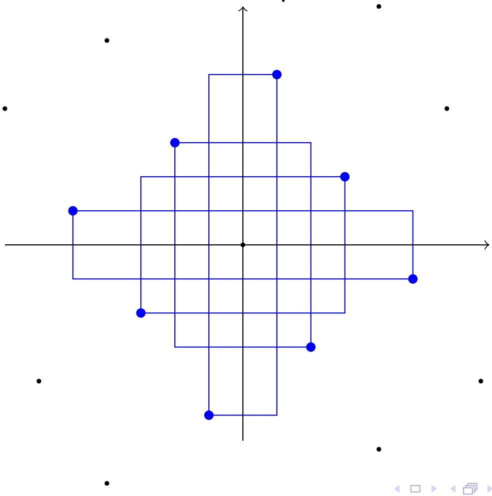
then

$$\mathrm{Box}(T) = [-|T|_1, |T|_1] \times \ldots \times [-|T|_s, |T|_s].$$

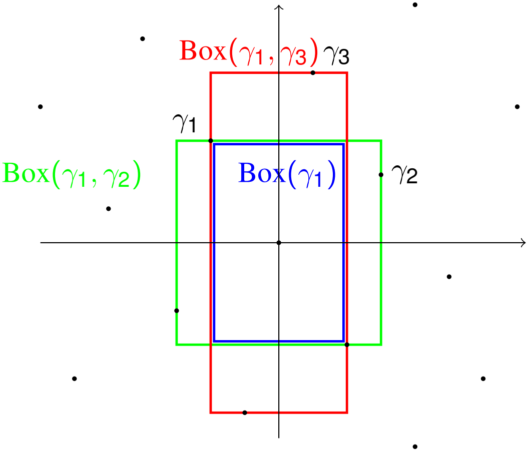
All boxes are symmetric with respect to the origin and axes-aligned.

A point γ in a lattice Γ is called a *relative (local) minimum* of the lattice Γ in the sense of Voronoi (or simply a *minimum*) if the $\mathrm{Box}(\gamma)$ is *free* (it contains no points of the lattice Γ different from its vertices and the origin).

2D example:



The $\mathrm{Box}(\gamma_1, \gamma_2)$ is called *extreme* if it is *free* and if, at the same time, it has on each of its faces at least one lattice point.
In other words it is impossible to extend this parallelepiped in any coordinate direction so that the resulting parallelepiped still contains no nonzero lattice points.



When we consider local minima or extreme parallelepipeds signs are not important for us. We can remove them.

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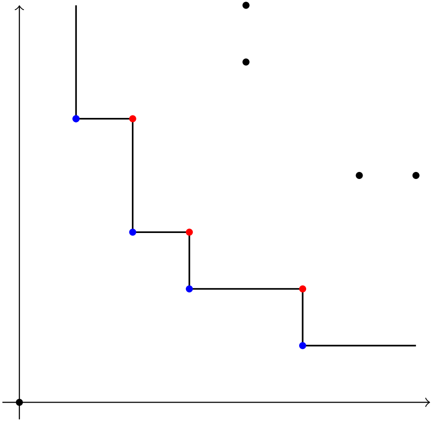
Instead of lattice Γ we can consider a set $|\Gamma| \subset \mathbb{R}^s$ where

$$|\Gamma| = \{(|x|, |y|, |z|) : (x, y, z) \in \Gamma\}.$$

As it was proved by Voronoi, we can consider a classical continued fraction as a sequence of local minima (halls) or extreme parallelepipeds (hills)



$\alpha \to \Gamma(\alpha) = \langle (1, 0), (\alpha, 1) \rangle$.

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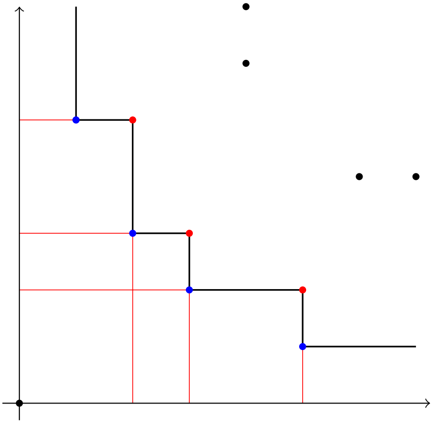

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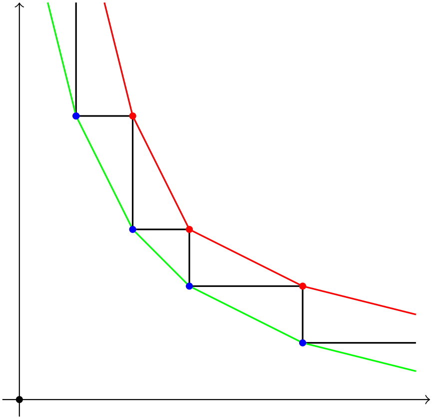
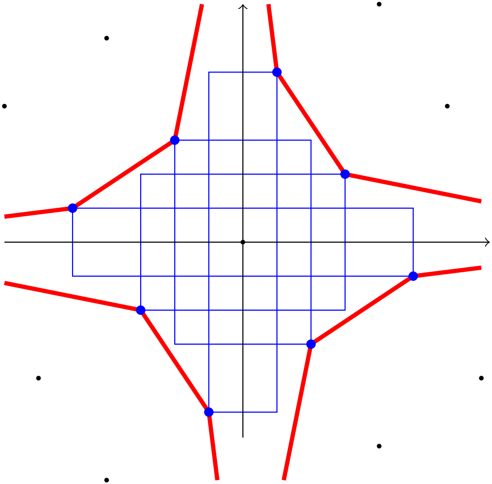


Local minima and Klein polyhedron: (in 2D case)

local minima=vertices of Klein polyhedron



In 3D case vertices of Klein polyhedron are always local minima, but converse is not true (Bykovski, 2006).
In other words local minima have more rich structure (they can hide bihind the faces of Klein polyhedron.

3D definitions

The $\mathrm{Box}(\gamma_1, \gamma_2, \gamma_3)$ is called *extreme* if it is *free* (it contains no lattice points other than the origin) and if, at the same time, it has on each of its faces at least one lattice point.

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A set of vectors (s.t. $v_i \neq v_j$) S in the lattice Γ is said to be *minimal* if the $\mathrm{Box}(S)$ contains no points of Γ except the origin. In particular, a minimal system of order 1 is a local minimum, minimal systems of order 3 gives extreme parallelepiped.

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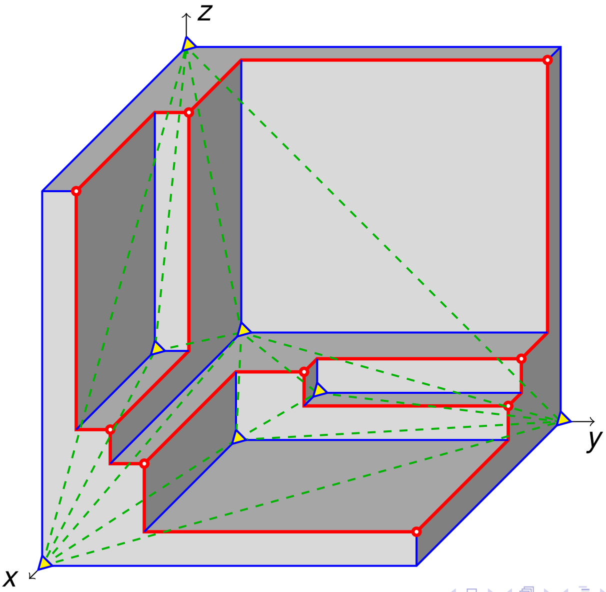
If $\{\gamma_1, \gamma_2\}$ is a minimal system of order 2 then γ_1 and γ_2 are *neighbours*.

Minkowski graph



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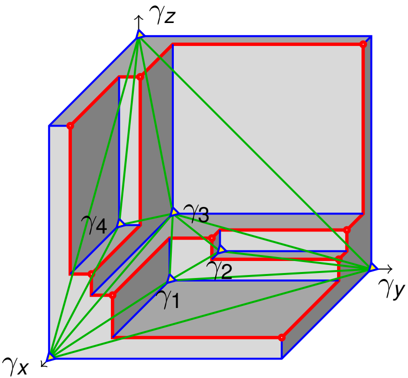
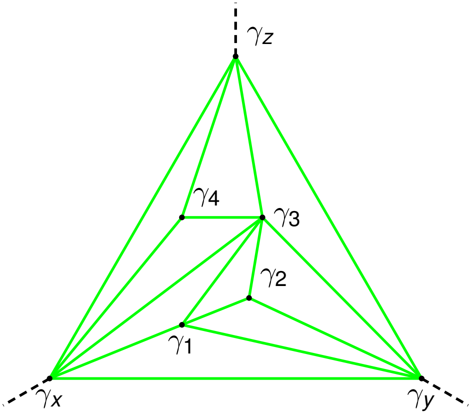
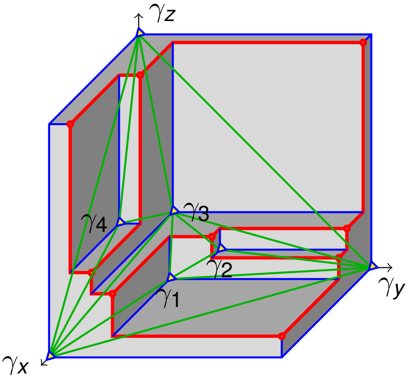
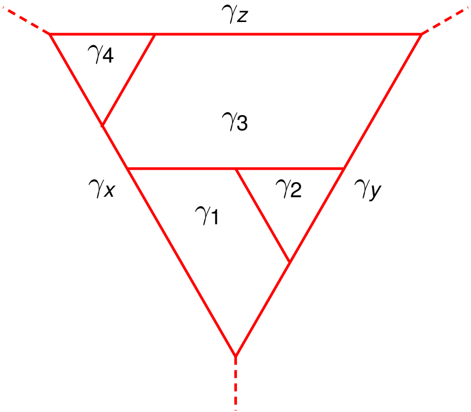




Voronoi (=Minkowski*) graph





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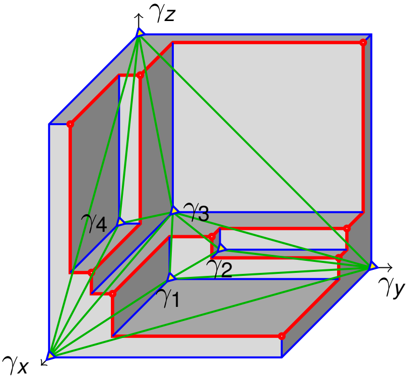

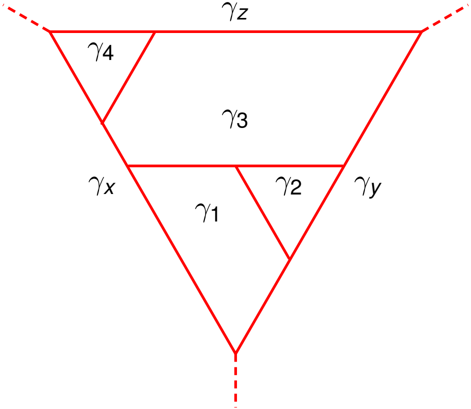

For more pictures and explanations see

O. Karpenkov, A. Ustinov "Geometry of Minkowski-Voronoi tessellations of the plane" (Journal of Number Theory, 2017).

Some reasons

Why do this objects are interesting and important?

- Good algorithms.
- Periodicity for algebraic lattices.
- “Vahlen’s theorem”.
- “Gauss measure”.
- Possibility to apply “hard” (analytical) methods based on Kloosterman sums.

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Voronoi considered chains of local minima.

They were able to do all calculations by hand

SYTA H., VAN DE WEYGAERT R. “Life and Times of Georgy Voronoï” (ArXiv e-prints, 2009).

“Markov asked Voronoï by telegraph to come from Warsaw to Petrograd. Markov invited Voronoï to his office and proposed him to calculate the unit for the cubic equation $r^3 = 23$. By artificial means, Markov had found for this example the unit

$$e = 2166673601 + 761875860r + 267901370r^2.$$

Voronoï calculated for three hours.

The period had 21 terms and in order to find the main unit it was necessary to multiply 21 expressions

$$-2+\rho,\ \frac{-11+2\rho+\rho^2}{15},\ \frac{-3-\rho+\rho^2}{4},\ \frac{-9+5\rho+\rho^2}{17},\ \frac{4-3\rho+\rho^2}{10},$$
$$\frac{1-\rho+\rho^2}{8},\ \frac{-2+\rho}{3},\ \frac{1+3\rho-\rho^2}{10},\ \frac{-5-\rho+\rho^2}{3},\ \frac{-1+\rho}{2},$$
$$\frac{-10+\rho+\rho^2}{11},\ -2+\rho,\ \frac{-11+2\rho+\rho^2}{15},\ \frac{1-\rho+\rho^2}{8},\ \frac{-2+\rho}{3},$$
$$\frac{1+3\rho-\rho^2}{5},\ \frac{-1+\rho}{2},\ \frac{-1+10\rho-\rho^2}{33},\ \frac{-11+7\rho+\rho^2}{20},$$
$$\frac{9-7\rho+2\rho}{31},\ \frac{-5-\rho+\rho^2}{6}.$$

Following this analysis, he found the unit

$$E = -41399 - 3160r + 6230r^2.$$

It turned out that $Ee = 1$. So, it was verified that the algorithm really worked."
Markov's unit one more time:

$$e = 2166673601 + 761875860r + 267901370r^2.$$

Theorem (Lagrange's Continued Fraction Theorem.)

The real roots of quadratic expressions with integral coefficients have periodic continued fractions.

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Two main examples (the beginning of *Markov spectrum*) are

$$\frac{1+\sqrt{5}}{2} = 2\cos\frac{2\pi}{5} = [1;1,\dots,1,\dots] = 1 + \cfrac{1}{1 + \ddots + \cfrac{1}{1+\dots}},$$

$$\sqrt{2} = 2\cos\frac{2\pi}{8} = [1;2,\dots,2,\dots] = 1 + \cfrac{1}{2 + \ddots + \cfrac{1}{2+\dots}}.$$

Some reasons

Periodicity

With quadratic irrational α we can associate a lattice $\Gamma(\alpha)$ with basis $(1, 1)$ and (α, β) where β is conjugate of α (second root of the same quadratic equation.)

Periodical continued fraction of α describes periodical structure of local minima of $\Gamma(\alpha)$.

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With cubic irrationality α (from totally real cubic field) we can associate 3D lattice with basis $(1,1,1)$, (α,β,γ), $(\alpha^2,\beta^2,\gamma^2)$, where β and γ are conjugates of α.

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Voronoi–Minkowski graph for such lattices is doubly periodic (totally real cubic field has 2 fundamental units).

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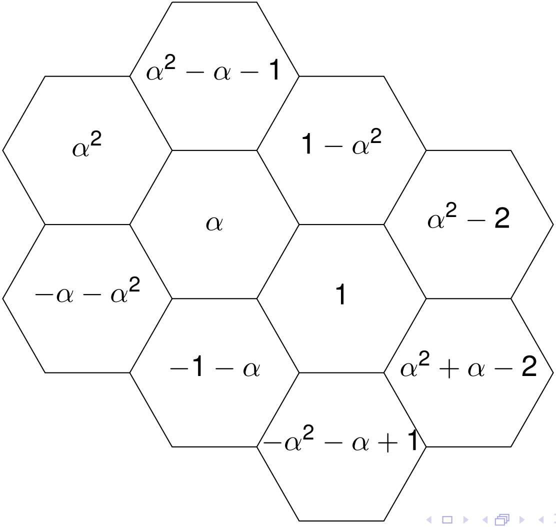
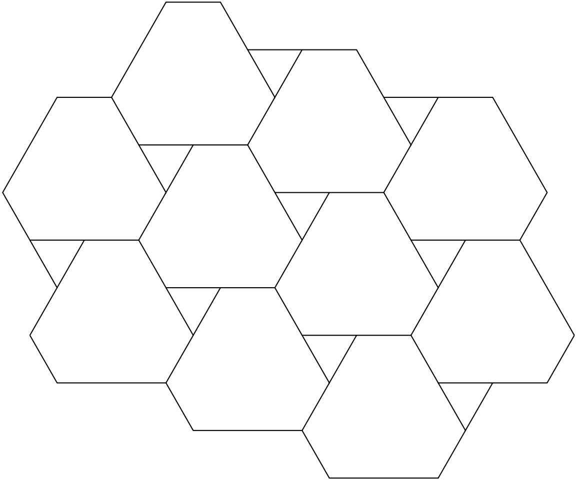
Voronoi–Minkowski graph for such lattices is doubly periodic (totally real cubic field has 2 fundamental units).
Two mains examples arise from cubic numbers $\alpha = 2\cos\frac{2\pi}{7}$ and $\alpha = 2\cos\frac{2\pi}{9}$ (associated with first two *extremal Davenport cubic forms*).

The Voronoi graph for $\alpha = 2\cos\frac{2\pi}{7}$



The Voronoi graph for $\alpha = 2\cos\frac{2\pi}{9}$



Some reasons

Vahlen's theorem

Denote by $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$ convergents to a given number $\alpha = [a_0; a_1, \dots, a_n, \dots]$.
Vahlen's theorem: for $p/q = p_{n-1}/q_{n-1}$ or $p/q = p_n/q_n$

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{2q^2}$$

can be translated to the lattice language. The equivalent statement: $\gamma_a = (a_1, a_2)$, $\gamma_b = (b_1, b_2)$ is a minimal system on lattice Γ, then

$$\min\{|a_1 a_2|, |b_1 b_2|\} \leqslant \frac{1}{2} \det \Gamma.$$

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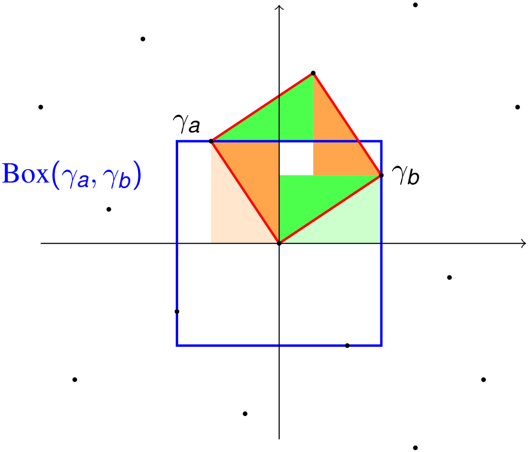
Vahlen's theorem has a stronger form:

$$|a_1 a_2| + |b_1 b_2| \leqslant \det \Gamma,$$

Vahlen theorem:

$$|a_1 a_2| + |b_1 b_2| \leqslant \det \Gamma.$$

$|a_1 a_2|$ is the orange area,
$|b_1 b_2|$ is the green area,
and their sum is less then the area of the red parallelogram.



Theorem (Avdeeva and Bykovskii, 2006)

If

$$\gamma_a = (a_1, a_2, a_3), \quad \gamma_b = (b_1, b_2, b_3), \quad \gamma_c = (c_1, c_2, c_3),$$

is a minimal system on lattice Γ*, then*

$$|a_1 a_2 a_3| + |b_1 b_2 b_3| + |c_1 c_2 c_3| \leqslant \det \Gamma.$$

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$$|a_1 a_2 a_3| + |b_1 b_2 b_3| + |c_1 c_2 c_3| \leqslant \det \Gamma.$$

This theorem can be regarded as a sharpening of the estimate

$$|a_1 a_2 a_3| + |b_1 b_2 b_3| + |c_1 c_2 c_3| \leqslant 3 \det \Gamma,$$

which follows from Minkowski's convex body theorem.
Similar statement is an open problem in dimension $d \geqslant 4$.

Some reasons

Theorem (Vahlen's theorem (local))

Let $\gamma_a = (a_1, a_2)$, $\gamma_b = (b_1, b_2)$ *be a minimal system on lattice* Γ, *then*

$$|a_1 a_2| + |b_1 b_2| \leqslant \det \Gamma.$$

Theorem (Markov-Hurwitz theorem (global))

Let Γ *be a 2D lattice then for some* $\gamma_a = (a_1, a_2) \in \Gamma$

$$|a_1 a_2| \leqslant \frac{1}{\sqrt{5}} \det \Gamma.$$

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Let $\gamma_a = (a_1, a_2)$, $\gamma_b = (b_1, b_2)$ *be a minimal system on lattice* Γ, *then*

$$\min\{|a_1 a_2|, |b_1 b_2|, |(a_1 + b_1)(a_2 + b_2)|\} \leqslant \frac{1}{\sqrt{5}} \det \Gamma.$$

Some reasons

Theorem (Davenport (global))

Let Γ *be a 3D lattice then for some* $\gamma_a = (a_1, a_2, a_3) \in \Gamma$

$$|a_1 a_2 a_3| \leqslant \frac{1}{7} \det \Gamma.$$

Open problem is make this statement local:
If

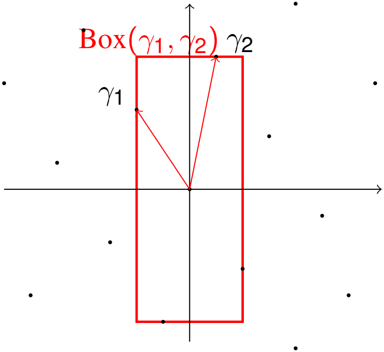
$$\gamma_a = (a_1, a_2, a_3), \quad \gamma_b = (b_1, b_2, b_3), \quad \gamma_c = (c_1, c_2, c_3),$$

is a minimal system on lattice Γ, then

$$\min\{|a_1 a_2 a_3|, |b_1 b_2 b_3|, |c_1 c_2 c_3|, \ldots\} \leqslant \frac{1}{7} \det \Gamma,$$

where "..." are four explicit linear combinations of γ_a, γ_b, γ_c with small coefficients.

In 2D case minimal couple $\gamma_1 = (a_1, b_1)$, $\gamma_2 = (a_2, b_2)$ is always a basis of a given lattice (Voronoi):



Some reasons

Gauss measure

We can associate with minimal system $\gamma_a = (a_1, a_2)$, $\gamma_b = (b_1, b_2)$ the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ with diagonal dominance: $|a_1| > |b_1|$, $|b_2| > |a_2|$.

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- permutation of rows or columns (renumbering of the vectors or of the coordinates axes);

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- changing the signs of all elements in a column (changing the direction of a coordinate axis);

Some reasons

Gauss measure

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- permutation of rows or columns (renumbering of the vectors or of the coordinates axes);
- changing the signs of all elements in a column (changing the direction of a coordinate axis);
- multiplication of a row by a nonzero number (rescaling one of the coordinate axes, possibly in combination with changing the orientation of this axis).

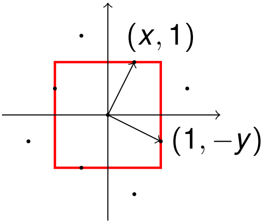
$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \to \begin{pmatrix} a_1^{-1} & 0 \\ 0 & b_2^{-1} \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & x \\ -y & 1 \end{pmatrix}$$

where $0 < x < 1$, $0 < y < 1$. $\mathrm{Box}(\gamma_1, \gamma_2) \to [-1, 1]^2$.

Gaussian measure

$$d\mu = \frac{dx\,dy}{(1+xy)^2} = \frac{dxdy}{\begin{vmatrix} 1 & x \\ -y & 1 \end{vmatrix}^2}$$

defined for $(x, y) \in [0, 1]^2$ describes typical behavior of classical continued fractions. From geometrical point of view this density function describes distribution of vectors from bases $\begin{pmatrix} 1 & x \\ -y & 1 \end{pmatrix}$ on the sides of unit square.



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In 2D case minimal couple $\gamma_1 = (a_1, b_1)$, $\gamma_2 = (a_2, b_2)$ is always a basis of a given lattice and $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \sim \begin{pmatrix} 1 & x \\ -y & 1 \end{pmatrix}$ where $0 < x < 1$, $0 < y < 1$.

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3D surprise (Minkowski): either minimal triple $\gamma_1 = (a_1, b_1, c_1)$, $\gamma_2 = (a_2, b_2, c_2)$, $\gamma_3 = (a_3, b_3, c_3)$ is a basis and corresponding matrix equivalent to

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or it is degenerate ($\det(\gamma_1, \gamma_2, \gamma_3) = 0$) and for some combination of signs

$$\gamma_1 \pm \gamma_2 \pm \gamma_3 = 0.$$

Open problem is to classify minimal systems in 4D.

The 3D analogue of Gaussian measure

$$d\mu = \frac{dx_2\, dx_3\, dy_1\, dy_3\, dz_1\, dz_2}{\begin{vmatrix} 1 & x_2 & \pm x_3 \\ -y_2 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{vmatrix}^3}$$

describes a distribution of basis vectors on some subset of $[0,1]^6$ (defined by some simple liner inequalities).

Kloosterman sums

Solutions (x, y) of the congruence

$$xy \equiv 1 \pmod{a}$$

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Applications

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This observation gives the way to study reduced bases in 2D lattices, continued fractions etc. Applications include:

- Gauss–Kuz'min statistics for rational numbers and quadratic irrationalities.
- Distribution of Frobenius numbers with 3 arguments.
- Distribution of free path lengths in 2D lattices (Lorenz gas).

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Propagating a solution we have 3 degrees of freedom (u, t, $z \in \mathbb{Z}$):

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = N \quad \Rightarrow \quad \begin{vmatrix} a & zb + ua \\ z^{-1}c + ta & * \end{vmatrix} = N$$

3-dimensional case ($3 \to 2$-reduction)

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $q = \det A \neq 0$ and

$$\begin{vmatrix} \multicolumn{2}{c}{\multirow{2}{*}{\Huge A}} & x_1 \\ & & x_2 \\ x_3 & x_4 & x_5 \end{vmatrix} = N.$$

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Linnik and Skubenko (1964) used this argument for studying distribution of points on a variety defined by equation $\det(x_{ij}) = N$ $(i, j = 1, 2, 3)$.

The goal

This tool allows to study Minkowski bases. The questions we can answer:
What is the average number of these bases?
How do the bases vector distributed?

The basic problem

Let $\ell(a/b)$ be a length of continued fraction expansion for a/b.

Theorem (Heilbronn, 1968)

$$\frac{1}{\varphi(N)}\sum_{\substack{1\leqslant a\leqslant N\\(a,N)=1}}\ell(a/N)=\frac{2\log 2}{\zeta(2)}\log N+O(\log^4\log N).$$

Theorem (Porter, 1975)

$$\frac{1}{\varphi(N)}\sum_{\substack{1\leqslant a\leqslant N\\(a,N)=1}}\ell(a/N)=\frac{2\log 2}{\zeta(2)}\log N+C_P+O(N^{-1/6+\varepsilon}),$$

$$C_P=\frac{2\log 2}{\zeta(2)}\left(\frac{3\log 2}{2}+2\gamma-2\frac{\zeta'(2)}{\zeta(2)}-1\right)-\frac{1}{2}.$$

These theorems count number of local minima, or number of extreme parallelepipeds, or the number of matrices of the form

$$\begin{pmatrix} a_1 & -b_1 \\ a_2 & b_2 \end{pmatrix}, \qquad \text{where} \quad a_2 \leqslant b_2, b_1 \leqslant a_1$$

with fixed determinant N.

Gauss–Kuz'min statistics

For rational $r = [a_0; a_1, \ldots, a_s]$ and real x, $y \in [0, 1]$ Gauss–Kuz'min statistics $\ell(x, y)$ can be defined as follows

$$\ell_{x,y}(r) = \left|\left\{1 \leqslant j \leqslant \ell + 1 : [0; a_j, \ldots, a_\ell] \leqslant x, [0; a_{j-1}, \ldots, a_1] \leqslant y\right\}\right|.$$

Theorem (The generalization of Porter's theorem)

$$\frac{1}{\varphi(N)} \sum_{\substack{1 \leqslant a \leqslant N \\ (a,N)=1}} \ell_{x,y}(a/N) = \frac{2\log(1+xy)}{\zeta(2)} \log N + C_P(x, y) + O(N^{-1/6+\varepsilon}).$$

The leading coefficient is a Gauss measure of corresponding box:

$$\log(1+xy) = \int_0^x \int_0^y \frac{d\alpha\, d\beta}{(1+\alpha\beta)^2} = \mu(Box = [0, x] \times [0, y])$$

The Gaussian measure

$$d\mu = \frac{dx\,dy}{\begin{vmatrix} 1 & x \\ -y & 1 \end{vmatrix}^2}$$

is a (right) Haar measure on quotient space $D_2(\mathbb{R})\backslash GL_2(\mathbb{R})$.

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The same measure describes behavior of Klein polyhedrons. The difference is in measure space. Measure space varies for different types of 3D continued fractions.

The main result is a 3D analogue of Porter's theorem.

Theorem (AU, 2015)

Average (over primitive lattices $\Lambda \subset \mathbb{Z}^3$ with $\det \Lambda = N$) number of elements in 3D continued fraction is

$$c_2 \log^2 N + c_1 \log N + c_0 + O(N^{-1/34+\varepsilon}).$$

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This theorem has a natural generalization on 3D Gauss — Kuz'min statistics. In this case the leading coefficient $c_2 = \mu(Box)$, $Box \subset [0, 1]^6$.

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The lattice with basis matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

is primitive iff

$$(X_1, X_2, X_3) = (Y_1, Y_2, Y_3) = (Z_1, Z_2, Z_3) = 1.$$

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(In 2D case we considered a/N such that $(a, N) = 1$.)

$$\Omega_1 = \left\{ \begin{pmatrix} a_1 & b_1 & -c_1 \\ -a_2 & b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{pmatrix}, \text{ where } \begin{matrix} b_1, c_1 \leqslant a_1; a_2, c_2 \leqslant b_2; \\ a_3, b_3 \leqslant c_3, \\ c_1 \leqslant b_1; \end{matrix} \right\}$$

$$\Omega_2 = \left\{ \begin{pmatrix} a_1 & b_1 & c_1 \\ -a_2 & b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{pmatrix}, \text{ where } \begin{matrix} b_1, c_1 \leqslant a_1; a_2, c_2 \leqslant b_2; \\ a_3, b_3 \leqslant c_3, \\ c_1 \leqslant b_1; a_2 + c_2 \geqslant b_2. \end{matrix} \right\}$$

$$\sum_{\substack{M \in \Omega_{1,2} \\ \det M = N}} 1 = ?$$

Thank you for your attention!