НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ
УНИВЕРСИТЕТ

Feynman checkers: external electromagnetic field and asymptotic properties

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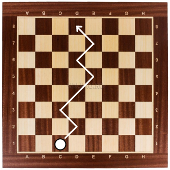
Plan

- Basic model (if needed)
- Model with external field
- Exact solution
- Continuum limit

Basic model

Fix $m \geq 0$ called the *mass* of an electron. Consider an infinite checkerboard made of squares $\varepsilon \times \varepsilon$. The checker moves to the diagonal-neighboring squares, either upwards-right or upwards-left. To each path s of the checker we assign a vector $a(s)$ as follows:

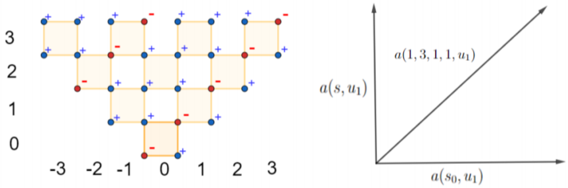
- Initially the vector is directed upwards and has unit length;
- After each turn of the checker it is rotated by 90° clockwise and multiplied by $m\varepsilon$;
- At the end of the motion the vector is shrinked by a factor of $(1 + m^2\varepsilon^2)^{\frac{t/\varepsilon - 1}{2}}$, where t/ε is the total number of moves.



(by V. Skopenkova)

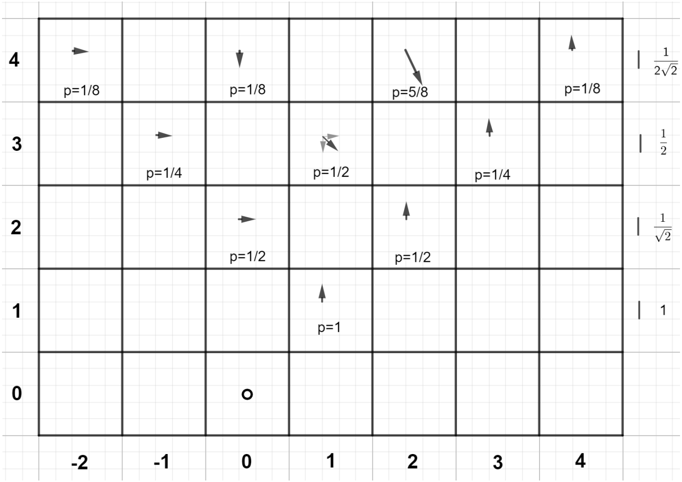
Basic model

Denote by $a(x, t, m, \varepsilon) := \sum_s a(s)$ the sum overall checker paths from the square $(0, 0)$ to the square (x, t), starting from the upwards-right move. The length square of the vector $a(x, t, m\varepsilon)$ is called *the probability to find an electron in the square* (x, t) *if it was emitted from the origin* and the vector itself is called *the arrow or the wave function.*



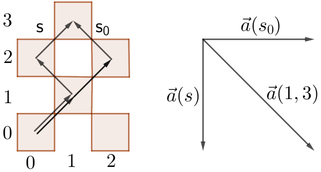
$$\bar{a}(1, 3, 1) = (\tfrac{1}{2}, -\tfrac{1}{2}),\ P(1, 3, 1) = \tfrac{1}{2}$$

Basic model

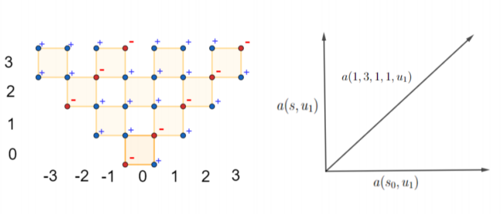


(by V. Skopenkova)

Model with external field



Basic model



Model with external field

For integer x/ε, t/ε the *homogeneous* field u_ε is given by the formula

$$u_\varepsilon(x+\varepsilon/2, t+\varepsilon/2) = \begin{cases} -1, & \text{if } (t-x)/4\varepsilon \in \mathbb{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

Model with external field

Definition 1.
Fix ε and $m \geq 0$. Consider the lattice $\varepsilon\mathbb{Z}^2 = \{(x,t) : x/\varepsilon, t/\varepsilon \in \mathbb{Z}\}$. Let u be a map from $\{(x,t) : x/\varepsilon, t/\varepsilon \in \mathbb{Z} + \frac{1}{2}\}$ into $\{\pm 1\}$. Denote by

$$a(x,t,m,\varepsilon,u) :=$$
$$(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2} i \sum_s (-im\varepsilon)^{\mathrm{turns}(s)} u(\frac{s_0+s_1}{2}) u(\frac{s_1+s_2}{2}) \dots u(\frac{s_{t/\varepsilon-1}+s_{t/\varepsilon}}{2})$$

the sum over all checker paths $s = (s_0, s_1, \dots, s_{t/\varepsilon})$, such that $s_0 = (0,0)$, $s_1 = (\varepsilon, \varepsilon)$, $s_{t/\varepsilon} = (x,t)$.
Denote

$$a_1(x,t,m,\varepsilon,u) := \operatorname{Re} a(x,t,m,\varepsilon,u),$$
$$a_2(x,t,m,\varepsilon,u) := \operatorname{Im} a(x,t,m,\varepsilon,u).$$

The value $|a(x,t,m,\varepsilon,u)|^2$ is called *the probability to find an electron of mass m at the point (x,t) on the lattice of step ε, if it was emitted from the point $(0,0)$ and moved in the field u.*

Model with external field

t \ x	-2	-1	0	1	2	3	4
4	$\frac{-1}{2\sqrt{2}}$		$\frac{2+i}{2\sqrt{2}}$		$\frac{-1}{2\sqrt{2}}$		$\frac{1}{2\sqrt{2}}i$
3		$\frac{-1}{2}$		$\frac{1+i}{2}$		$\frac{-1}{2}i$	
2			$\frac{-1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}i$		
1				$-i$			

Values of $a(x, t, 1, 1, u_1)$ in homogeneous field for small x and t.

Exact solution

Denote by $\delta_2(b)$ the remainder of b after division by 2.

Proposition (F.O., 2022)

For each real $m \geq 0$ and integer $\xi, \eta \geq 0$ the following equalities hold:

$$a_1(\xi - \eta + 1, \xi + \eta + 1, m, 1, u_1) =$$

$$= (-1)^{\xi+1} \frac{m(1+m^2)^{\delta_2(\xi(\eta+1))}}{(1+m^2)^{\frac{\xi+\eta}{2}}} \sum_{j=0}^{\lfloor \frac{\xi}{2} \rfloor} \binom{\lfloor \frac{\xi}{2} \rfloor}{j} \binom{\lfloor \frac{\eta-1}{2} \rfloor}{j} (1-(1+m^2)^2)^j;$$

$$a_2(\xi - \eta + 1, \xi + \eta + 1, m, 1, u_1) =$$

$$= \frac{(-1)^{\xi+1}}{(1+m^2)^{\frac{\xi+\eta}{2}}} \sum_{j=0}^{\lfloor \frac{\xi}{2} \rfloor} \left(\binom{\lfloor \frac{\eta}{2} \rfloor}{j} (1+m^2)^{\delta_2(\xi\eta)} - \right.$$

$$\left. - \binom{\lfloor \frac{\eta-1}{2} \rfloor}{j} (1+m^2)^{\delta_2(\xi(\eta+1))} \right) \binom{\lfloor \frac{\xi}{2} \rfloor}{j} (1-(1+m^2)^2)^j.$$

Exact solution in terms of Hypergeometric functions.

For integer a, b, c, where $b \leq 0$, the polynomial.

$$ {}_2F_1\left(a, b; c; z\right) = 1 + \sum_{k=1}^{\infty} \prod_{l=0}^{k-1} \frac{(a+l)(b+l)}{(1+l)(c+l)} z^k, $$

is called *Gauss Hypergeometric function*.

Proposition (F.O., 2022)

Denote $z = 1 - (1+m^2)^2$. Then for each real $m \geq 0$ and integer $\xi, \eta \geq 0$ the following equalities hold:

$$ a_1(\xi - \eta + 1, \xi + \eta + 1, m, 1, u_1) = $$
$$ = (-1)^{\xi+1} m(1+m^2)^{-\frac{\xi+\eta}{2} + \delta_2((1+\eta)\xi)} \cdot {}_2F_1\left(-\left\lfloor \frac{\eta - 1}{2} \right\rfloor, -\left\lfloor \frac{\xi}{2} \right\rfloor; 1; z\right). $$

Remark

There is a similar formula for $a_2(x, t, m, 1, u_1)$.

Known formula for the basic model

Theorem (Folklore)

For each real $m \geq 0$ and integer $\xi, \eta \geq 0$ the following equalities hold:

$$a_1(\xi - \eta + 1, \xi + \eta + 1, m, 1) = m(1 + m^2)^{-\frac{\xi+\eta}{2}} \cdot {}_2F_1\left(-\xi, 1 - \eta; 1; -m^2\right);$$

$$a_2(\xi - \eta + 1, \xi + \eta + 1, m, 1) = -\frac{\xi}{2} m^2 (1 + m^2)^{-\frac{\xi+\eta}{2}} \cdot {}_2F_1\left(1 - \xi, 1 - \eta; 2; -m^2\right).$$

Continuum limit

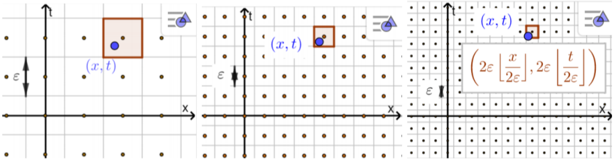
Theorem (F.O., 2022)

Let u_ε be the homogeneous electromagnetic field. Then for each $m > 0$ and $|x| < t$ we have:

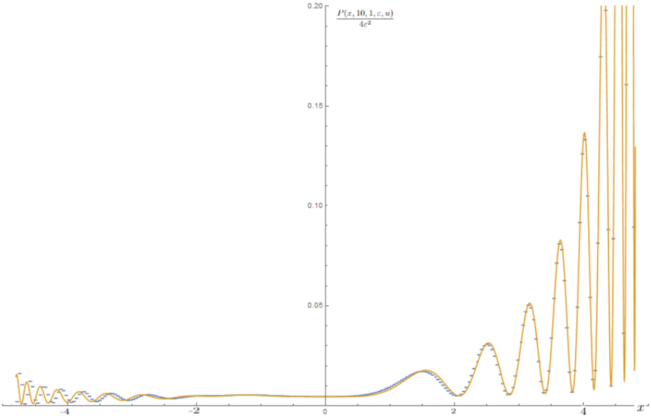
$$\lim_{\varepsilon\searrow 0} \frac{1}{2\varepsilon} a_1\left(4\varepsilon\left\lfloor\frac{x}{4\varepsilon}\right\rfloor, 4\varepsilon\left\lfloor\frac{t}{4\varepsilon}\right\rfloor, m, \varepsilon, u_\varepsilon\right) = \frac{m}{2} J_0\left(m\sqrt{\frac{t^2-x^2}{2}}\right),$$

$$\lim_{\varepsilon\searrow 0} \frac{1}{2\varepsilon} a_2\left(4\varepsilon\left\lfloor\frac{x}{4\varepsilon}\right\rfloor, 4\varepsilon\left\lfloor\frac{t}{4\varepsilon}\right\rfloor, m, \varepsilon, u_\varepsilon\right) = -\frac{m}{\sqrt{2}}\sqrt{\frac{t+x}{t-x}} J_1\left(m\sqrt{\frac{t^2-x^2}{2}}\right).$$

Here $J_0(z) := \sum_{j=0}^{\infty}(-1)^j \frac{(z/2)^{2j}}{(j!)^2}$ and $J_1(z) := \sum_{j=0}^{\infty}(-1)^j \frac{(z/2)^{2j+1}}{(j!)(j+1)!}$ are *Bessel functions* of the first kind of orders 0 and 1 respectively.



Continuum limit



Continuum limit in basic model and mass renormalization.

Theorem (Skopenkov-Ustinov 2022, Lvov 2022, Narlikar 1971)

Assume $m, \varepsilon > 0$, $|x| < t$, where $x/2\varepsilon, t/2\varepsilon \in \mathbb{Z}$. Then

$$\lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} a_1 \left(2\varepsilon \left\lfloor \frac{x}{2\varepsilon} \right\rfloor, 2\varepsilon \left\lfloor \frac{t}{2\varepsilon} \right\rfloor, m, \varepsilon \right) = J_0 \left(m\sqrt{t^2 - x^2} \right)$$

$$\lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} a_2 \left(2\varepsilon \left\lfloor \frac{x}{2\varepsilon} \right\rfloor, 2\varepsilon \left\lfloor \frac{t}{2\varepsilon} \right\rfloor, m, \varepsilon \right) = \sqrt{\frac{t+x}{t-x}} J_1 \left(m\sqrt{t^2 - x^2} \right).$$

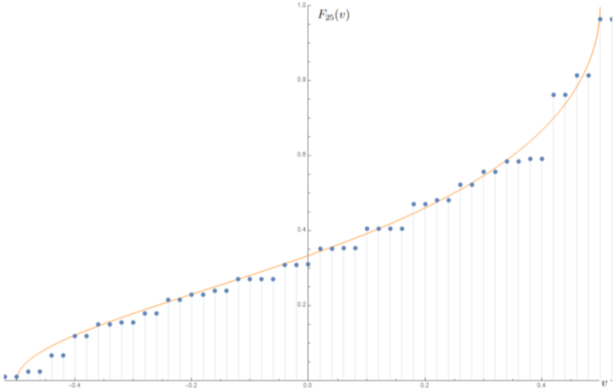
Remark

The relation between the arguments of the Bessel functions in these models is given by mass renormalization:

$$m = \frac{m_0}{\sqrt{2}},$$

where m is the mass in the model with the field, and m_0 is the one in the model without field.

Large time limit



Large time limit

Theorem (F.O., 2022)

For each real $m, \varepsilon > 0$ and each real v the following equality holds

$$\lim_{\substack{t\to\infty \\ t\in\varepsilon\mathbb{Z}}} \sum_{\substack{x\leq vt \\ x\in\varepsilon\mathbb{Z}}} P(x,t,m,\varepsilon,u_\varepsilon) = F(v) := \begin{cases} 0, & \text{if } v < -\frac{1}{1+m^2\varepsilon^2}; \\ \frac{1}{\pi}\arccos\frac{1-(1+m^2\varepsilon^2)^2 v}{(1+m^2\varepsilon^2)(1-v)}, & \text{if } |v| \leq \frac{1}{1+m^2\varepsilon^2}; \\ 1, & \text{if } v > \frac{1}{1+m^2\varepsilon^2}. \end{cases}$$

Theorem (Grimmet-Janson-Scudo, 2004)

For each real $m, \varepsilon > 0$ and each real v the following equality holds

$$\lim_{\substack{t\to\infty \\ t\in\varepsilon\mathbb{Z}}} \sum_{\substack{x\leq vt \\ x\in\varepsilon\mathbb{Z}}} P(x,t,m,\varepsilon) = F(v) := \begin{cases} 0, & \text{if } v < -\frac{1}{\sqrt{1+m^2\varepsilon^2}}; \\ \frac{1}{\pi}\arccos\frac{1-(1+m^2\varepsilon^2)v}{\sqrt{1+m^2\varepsilon^2}(1-v)}, & \text{if } |v| \leq \frac{1}{\sqrt{1+m^2\varepsilon^2}}; \\ 1, & \text{if } v > \frac{1}{\sqrt{1+m^2\varepsilon^2}}. \end{cases}$$

Remark

Here the relation between m_0 and m is given by the formula $(1+m^2\varepsilon^2)^2 = 1+m_0^2\varepsilon^2$. However, tending ε to 0 we obtain exactly the relation from the continuum limit case.

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Thanks for your attention!