Uniform approximation of the wave function by Airy function

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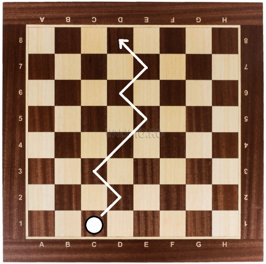
Overview

1. Definition of the Feynman checkers
2. Uniform approximation
3. Proof Outline

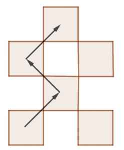
Informal definition

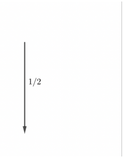


First idea: path $\mapsto$ complex number

Informal definition

path $\mapsto$ complex number

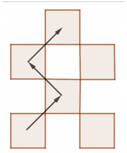


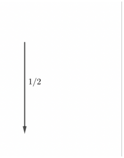


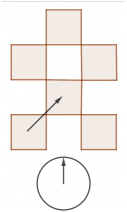
length $= \frac{1}{2^{(t-1)/2}}$,
where $t =$
number of moves

Informal definition

path $\mapsto$ complex number







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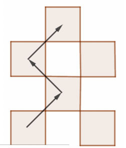

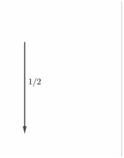

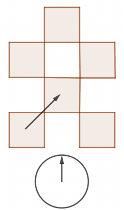

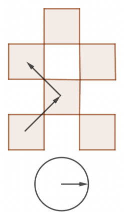

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Informal definition

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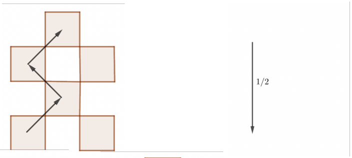

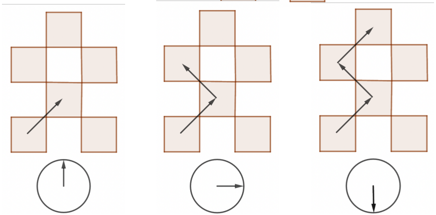

length $= \frac{1}{2^{(t-1)/2}}$,
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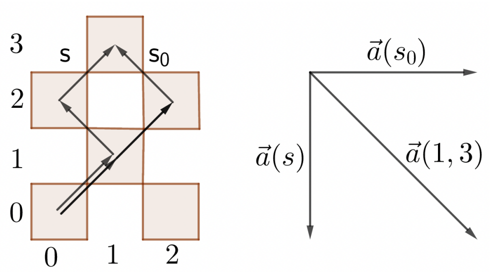
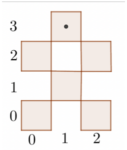
$\leftarrow$ direction

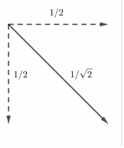


Second idea: square $\mapsto$ complex number

Informal definition

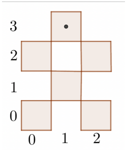
square $\mapsto$ complex number

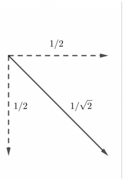


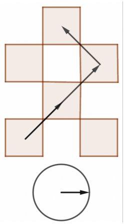


vector of a square $=$
sum of vectors
of paths

square $\mapsto$ complex number



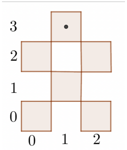


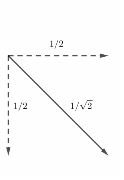


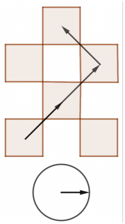
vector of a square =
sum of vectors
of paths

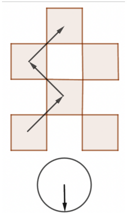
Informal definition

square $\mapsto$ complex number









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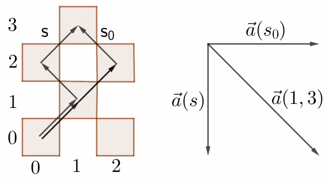
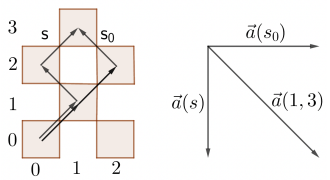
Definition (R.Feynman, 1950s)



$$a(x,t) := 2^{(1-t)/2}\, i \sum_s (-i)^{\mathrm{turns}(s)}$$

is the sum over all checker paths s from $(0,0)$ to (x,t) with the first step to $(1,1)$, where *turns*(s) is the number of turns in s.



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The length square of the resulting vector is the **probability to find an electron in the square** (x,t)**, if it was emitted from** $(0,0)$:

$$P(1,3) = |a(1,3)|^2 = 1/2.$$

Definition (R.Feynman, 1950s)

- **Wave function**:
$$a(x,t) = 2^{(1-t)/2}\, i \sum_s (-i)^{\mathrm{turns}(s)};$$
$$a_1(x,t) := \mathrm{Re}(a(x,t)), \qquad a_2(x,t) := \mathrm{Im}(a(x,t));$$
- **Probability** to find an electron at (x,t), if it was emitted from $(0,0)$:
$$P(x,t) := |a(x,t)|^2.$$

Integral form of wave functions

Lemma (Fourier integral representation of the wave functions)

For every integer x and t such that $x+t$ is odd we have

$$a_1(x,t+1) = \frac{(-1)^{(x-t+1)/2}}{2\pi} \int_{-\pi}^{\pi} \frac{e^{itL(u,x/t)}}{\sqrt{1+\cos^2(u)}} du;$$

for every integer x and t such that $x+t$ is even we have

$$a_2(x+1,t+1) = \frac{(-1)^{(x-t)/2}}{2\pi} \int_{-\pi}^{\pi} \left(1 + \frac{\cos u}{\sqrt{1+\cos^2 u}}\right) e^{itL(u,x/t)} du,$$

where

$$L(u,v) := uv - \arcsin\left(\frac{\sin u}{\sqrt{2}}\right).$$

$L(u, v)$

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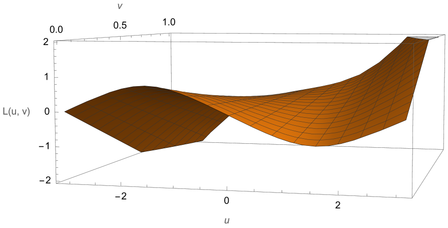

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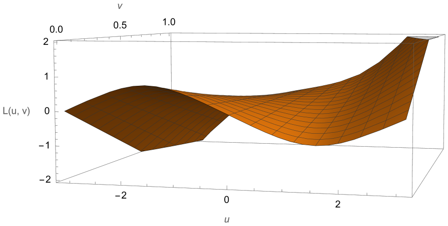

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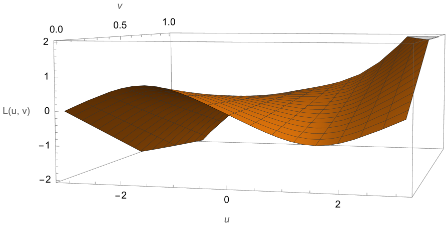

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$$\frac{\partial L}{\partial u} = v - \frac{\cos u}{\sqrt{1+\cos^2 u}}$$

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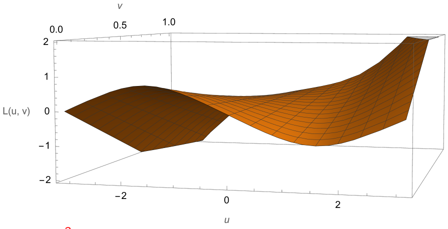

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$$\frac{\partial L}{\partial u} = v - \frac{\cos u}{\sqrt{1+\cos^2 u}} \Rightarrow \frac{\partial L}{\partial u} > 0 \text{ for } v > \frac{1}{\sqrt{2}}$$

Airy function:

$$\mathrm{Ai}(\lambda) := \frac{1}{\pi}\int_0^\infty \cos\left(\lambda p + \frac{p^3}{3}\right)\, dp\ .$$

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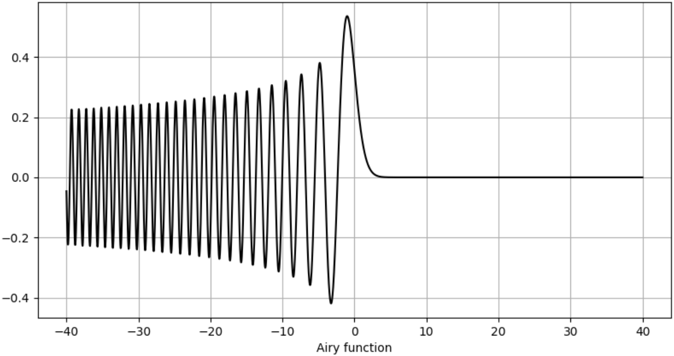

Main Result: Uniform approximation

Theorem (Uniform approximation of the wave functions)

For every integer x and t such that $|x| < t/\sqrt{2}$ and $x+t$ is odd we have

$$a_1(x, t+1) = (-1)^{\frac{|x|-t+1}{2}} \left(\frac{4\theta(x/t)}{1-2(x/t)^2}\right)^{\frac{1}{4}} \left(\frac{1}{t}\right)^{\frac{1}{3}} \mathrm{Ai}\left(-\theta(x/t)t^{\frac{2}{3}}\right) + O\left(\frac{1}{t}\right),$$

for every integer x and t such that $|x| < t/\sqrt{2}$ and $x+t$ is even we have

$$a_2(x+1, t+1) = (-1)^{\frac{|x|-t}{2}} \sqrt{\frac{t+x}{t-x}} \left(\frac{4\theta(x/t)}{1-2(x/t)^2}\right)^{\frac{1}{4}} \left(\frac{1}{t}\right)^{\frac{1}{3}} \mathrm{Ai}\left(-\theta(x/t)t^{\frac{2}{3}}\right) + O\left(\frac{1}{t}\right),$$

where

$$\theta(v) := \left(\frac{3}{2}\left(-|v| \arccos\left(\frac{|v|}{\sqrt{1-v^2}}\right) + \arccos\left(\frac{1}{\sqrt{2-2v^2}}\right)\right)\right)^{2/3}.$$

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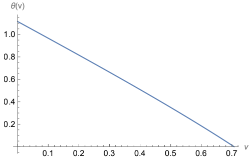

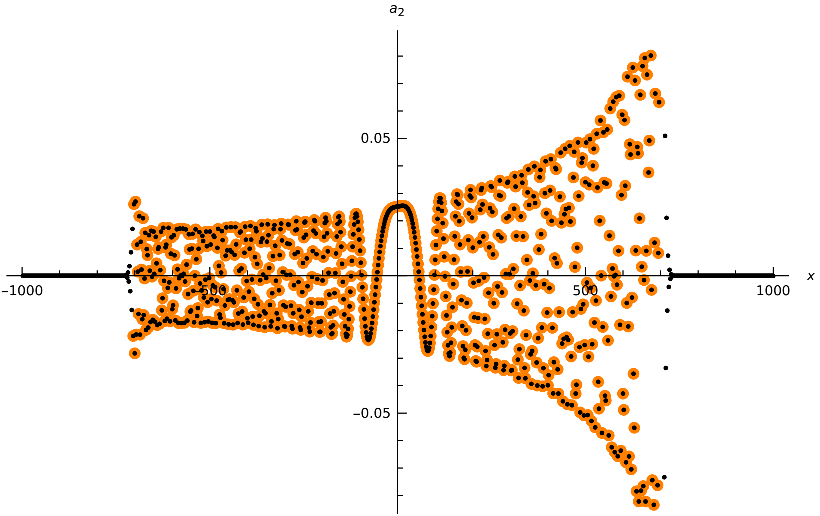

Figure: The plot of $a_2(x, 1000)$ for x even is shown in black and the approximation is shown in orange.

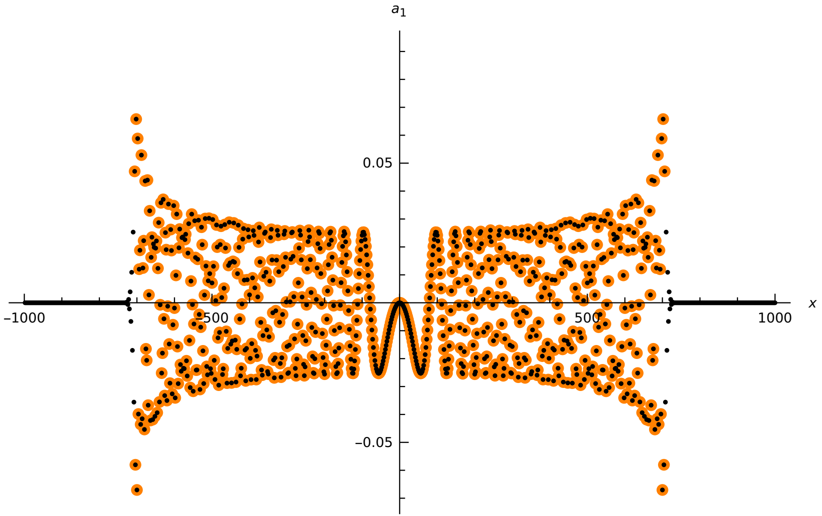

Figure: The plot of $a_1(x, 1000)$ for x even is shown in black and the approximation is shown in orange.

Theorem (Smooth change to cubic polynomial)

Let $f \colon [-U, U] \times [0, A] \to \mathbb{R}$ *be a real analytic function such that*

1. $f'_u(0,0) = 0$, $f''_{uu}(0,0) = 0$, $f'''_{uuu}(0,0) \neq 0$, $f''_{u\alpha}(0,0) \neq 0$;
2. $f(-u, \alpha) = -f(u, \alpha)$ *for each* $u \in [-U, U]$ *and* $\alpha \in [0, A]$;
3. *for each* $\alpha \in (0, A]$ *there are precisely two solutions* $\pm u_0(\alpha) \in [-U, U]$ *of the equation* $f'_u(u, \alpha) = 0$, *where* $u_0(\alpha) \in (0, U)$, *and we have* $f''_{uu}(u_0(\alpha), \alpha) > 0$;
4. *for* $\alpha = 0$ *there is only one solution* $u_0(0) = 0$ *of the equation* $f'_u(u, \alpha) = 0$.

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4. *for* $\alpha = 0$ *there is only one solution* $u_0(0) = 0$ *of the equation* $f'_u(u,\alpha) = 0$.

Then there exists an infinitely differentiable function $q\colon [-U, U] \times [0, A] \to \mathbb{R}$ *such that* $q'_u(u,\alpha) > 0$, $q(-u,\alpha) = -q(u,\alpha)$ *identically, and*

$$\boxed{f(u,\alpha) = \frac{q(u,\alpha)^3}{3} - \theta(\alpha) q(u,\alpha),}$$

where

$$\theta(\alpha) = \left(-\frac{3}{2} f(u_0(\alpha), \alpha)\right)^{2/3}.$$

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2. $f(-u,\alpha)=-f(u,\alpha)$ *for each* $u\in[-U,U]$ *and* $\alpha\in[0,A]$;
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$$f(u,\alpha)=\frac{q(u,\alpha)^3}{3}-\theta(\alpha)q(u,\alpha),$$

where

$$\theta(\alpha)=\left(-\frac{3}{2}f(u_0(\alpha),\alpha)\right)^{2/3}.$$

Moreover,

$$q(u_0(\alpha),\alpha)=\theta(\alpha)^{1/2},$$

$$q'_u(u_0(\alpha),\alpha)=h(\alpha):=\begin{cases}\dfrac{f''_{uu}(u_0(\alpha),\alpha)^{1/2}}{(-12f(u_0(\alpha),\alpha))^{1/6}}, & \text{if } 0<\alpha\leqslant A,\\ (\frac{1}{2}f'''_{uuu}(0,0))^{1/3}, & \text{if } \alpha=0.\end{cases}$$

Theorem (General uniform approximation by Airy function)

Let $f:[-U,U]\times[0,A]\to\mathbb{R}$ *be a real analytic function that satisfies conditions 1-4,* $g\in C^{\infty}([-U,U])$ *be even function, and* $t\in\mathbb{R}_{>0}$. *Then for each* $\alpha\geqslant 0$

$$\int_{-U}^{U} g(u)e^{itf(u,\alpha)}du = \frac{2\pi}{t^{1/3}}\frac{g(u_0(\alpha))}{h(\alpha)}\,\mathrm{Ai}\left(-\left(-\frac{3}{2}f(u_0(\alpha),\alpha)\cdot t\right)^{2/3}\right)+O_{f,g}\left(\frac{1}{t}\right),$$

where $u_0(\alpha)$ *is defined in conditions 3 and 4, and* $h(\alpha)$ *is defined in Theorem above.*

Proof of general uniform approximation theorem

Here we follow [3, Theorem 2, Appendix A] First we integrate by substitution $q = q(u, \alpha)$

$$I = \int_{-U}^{U} g(u) e^{itf(u,\alpha)} du$$

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$$I_1(t,\alpha) := \frac{g(u_0(\alpha))}{q'_u(u_0(\alpha),\alpha)} \int_{-q(U,\alpha)}^{q(U,\alpha)} e^{it(q^3/3 - \theta(\alpha)q)} dq,$$

$$I_2(t,\alpha) := \int_{-q(U,\alpha)}^{q(U,\alpha)} \left(g(u(q,\alpha)) u'_q(q,\alpha) - \frac{g(u_0(\alpha))}{q'_u(u_0(\alpha),\alpha)} \right) e^{it(q^3/3 - \theta(\alpha)q)} dq.$$

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$I_1(t,\alpha) \to$ main contribution

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$$I_2(t,\alpha) := \int_{-q(U,\alpha)}^{q(U,\alpha)} \left(g(u(q,\alpha)) u'_q(q,\alpha) - \frac{g(u_0(\alpha))}{q'_u(u_0(\alpha),\alpha)} \right) e^{it(q^3/3-\theta(\alpha)q)} dq.$$

$I_1(t,\alpha) \to$ main contribution ; $I_2(t,\alpha) = O_{f,g}(1/t)$.

Proof of general uniform approximation theorem

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Proposition

$$I_1(t,\alpha) = \frac{g(u_0(\alpha))}{h(\alpha)} \frac{2\pi}{t^{1/3}} \operatorname{Ai}\left(-\left(-\frac{3}{2}f(u_0(\alpha),\alpha)\cdot t\right)^{2/3}\right) + O_{f,g}\left(\frac{1}{t}\right).$$

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Lemma

For each θ and $Q > 0$ such that $Q^2 > \theta$ we have

$$\left|\int_Q^\infty \cos\left(\left(\frac{q^3}{3} - \theta q\right) t\right) dq\right| \leqslant \frac{2}{t(Q^2-\theta)}.$$

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Proof of Lemma.

Integration by parts

$$\cos\left(\frac{tq^3}{3} - \theta tq\right) = \frac{2tq\sin(tq^3/3-\theta tq)}{(tq^2-\theta t)^2} + \frac{d}{dq}\left(\frac{\sin(tq^3/3-\theta tq)}{tq^2-\theta t}\right).$$

□

$$I_2(t,\alpha) := \int_{-q(U,\alpha)}^{q(U,\alpha)} \underbrace{\left(g(u(q,\alpha))u'_q(q,\alpha) - \frac{g(u_0(\alpha))}{q'_u(u_0(\alpha),\alpha)} \right)}_{:=\phi(q,\alpha)} e^{it(q^3/3-\theta(\alpha)q)} dq.$$

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Proof of Proposition.

Use Hadamard's lemma with parameter to prove the smoothness of $G(q,\alpha) := \phi(q,\alpha)/(q^2 - \theta(\alpha))$

$$I_2(t,\alpha) := \int_{-q(U,\alpha)}^{q(U,\alpha)} G(q,\alpha)(q^2 - \theta(\alpha))e^{it(q^3/3-\theta(\alpha)q)} dq,$$

then integration by parts. □

Theorem (Smooth change to cubic polynomial)

Let $f\colon [-U,U]\times[0,A]\to\mathbb{R}$ be a real analytic function such that

1. $f'_u(0,0)=0$, $f''_{uu}(0,0)=0$, $f'''_{uuu}(0,0)\neq 0$, $f''_{u\alpha}(0,0)\neq 0$;
2. $f(-u,\alpha)=-f(u,\alpha)$ *for each* $u\in[-U,U]$ *and* $\alpha\in[0,A]$;
3. *for each* $\alpha\in(0,A]$ *there are precisely two solutions* $\pm u_0(\alpha)\in[-U,U]$ *of the equation* $f'_u(u,\alpha)=0$, *where* $u_0(\alpha)\in(0,U)$, *and we have* $f''_{uu}(u_0(\alpha),\alpha)>0$;
4. *for* $\alpha=0$ *there is only one solution* $u_0(0)=0$ *of the equation* $f'_u(u,\alpha)=0$.

Then there exists an infinitely differentiable function $q\colon[-U,U]\times[0,A]\to\mathbb{R}$ *such that* $q'_u(u,\alpha)>0$, $q(-u,\alpha)=-q(u,\alpha)$ *identically, and*

$$f(u,\alpha)=\frac{q(u,\alpha)^3}{3}-\theta(\alpha)q(u,\alpha),$$

where

$$\theta(\alpha)=\left(-\frac{3}{2}f(u_0(\alpha),\alpha)\right)^{2/3}.$$

Moreover,

$$q(u_0(\alpha),\alpha)=\theta(\alpha)^{1/2},$$

$$q'_u(u_0(\alpha),\alpha)=h(\alpha):=\begin{cases}\dfrac{f''_{uu}(u_0(\alpha),\alpha)^{1/2}}{(-12f(u_0(\alpha),\alpha))^{1/6}}, & \text{if } 0<\alpha\leqslant A,\\ \left(\frac{1}{2}f'''_{uuu}(0,0)\right)^{1/3}, & \text{if } \alpha=0.\end{cases}$$

Let

$$F(u, q, \alpha) := f(u, \alpha) - (q^3/3 - \theta(\alpha)q).$$

Consider $F(u, q, \alpha)$ as the cubic polynomial $F_{u\alpha}(q)$ in q (with parameters u and α). We choose an appropriate root $q = q(u, \alpha)$ of $F_{u\alpha}(q)$ for each $(u, \alpha) \in [-U, U] \times [0, A]$:

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$$q(u,\alpha) := \begin{cases} \text{the minimal root of } F_{u\alpha}(q), & \text{if } u \in [-U; -u_0(\alpha)], \\ \text{the middle root of } F_{u\alpha}(q), & \text{if } u \in (-u_0(\alpha); u_0(\alpha)), \\ \text{the maximal root of } F_{u\alpha}(q), & \text{if } u \in [u_0(\alpha), U]. \end{cases}$$

Proof of smooth change to cubic polynomial theorem

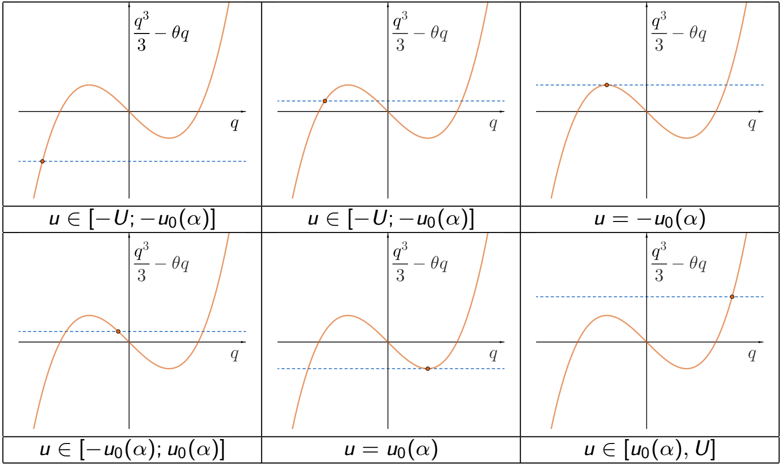

Figure: The choice of the root q of $F_{u\alpha}(q)$ for different u
(Blue lines are $f(u, \alpha) - (q^3/3 - \theta(\alpha)q) = 0$ for different u)

Recall

$$F(u, q, \alpha) = f(u, \alpha) - (q^3/3 - \theta(\alpha)q).$$

We need to prove the smoothness of $q(u, \alpha)$. There are three types of points in $[-U, U] \times [0, A]$:

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Case 2: $f'_u(u,\alpha) = 0$, $f''_{uu}(u,\alpha) \neq 0$

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Case 3: $f'_u(u, \alpha) \neq 0$ — all other points; apply the implicit function theorem to $F(u, v, \alpha)$.

Case 1: $f'_u(u,\alpha)=0,\ f''_{uu}(u,\alpha)=0$

Lemma 1 ([4, Ch. VI, Lemma 2.1])

Let $f(u,\alpha)$ be a function holomorphic in both variables for small $|u|$ and $|\alpha|$ satisfying condition 1 in Theorem. Then there is $R>0$ and a function $v(z)$ holomorphic for $|z|<R$ such that for $0<|\alpha|<R$ the equation $f'_u(u,\alpha)=0$ has precisely two solutions in the region $|u|<R$, given by $\{u_1(\alpha),u_2(\alpha)\}=\{v(\sqrt{\alpha}),v(-\sqrt{\alpha})\}$.

Lemma 2 ([4, Ch. VI, Lemma 2.2])

There exist functions $A(\alpha)$, $B(\alpha)$ holomorphic for small $|\alpha|$ such that

$$A(\alpha)=\frac{1}{2}\left(f(u_1(\alpha),\alpha)+f(u_2(\alpha),\alpha)\right),$$

$$B(\alpha)^3=\left(\frac{3}{4}(f(u_2(\alpha),\alpha)-f(u_1(\alpha),\alpha))\right)^2.$$

Lemma 3 ([4, Ch. VI, Lemma 2.3])

There exist R_1, $R_2>0$ and a function $q(u,\alpha)$, which is holomorphic for $|u|<R_1$, $|\alpha|<R_2$ and satisfies the equation

$$f(u,\alpha)=\frac{q(u,\alpha)^3}{3}-B(\alpha)q(u,\alpha)+A(\alpha).$$

Proof of smooth change to cubic polynomial theorem

Case 2: $f'_u(u,\alpha) = 0$, $f''_{uu}(u,\alpha) \neq 0$

Lemma (Morse lemma with a parameter [4, Ch. II, Lemma 3.3])

Let $S(r,\alpha)$ be a real-valued function such that

1. $S(r,\alpha) \in C^\infty(U \times V)$, *where* $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ *are neighbourhoods of* r_0, α_0 *respectively;*
2. $S'_r(r_0,\alpha) = 0$ *for each* $\alpha \in V$, *and* $S'_r(r,\alpha) \neq 0$ *for each* $r \in U \backslash \{r_0\}$, $\alpha \in V$;
3. $\det S''_{rr}(r_0,\alpha) \neq 0$ *for each* $\alpha \in V$.

Then there exist neighbourhoods V_0, U_0, W *of the points* $\alpha = \alpha_0$, $r = r_0$, $y = 0$ *respectively, and a function* $r = \Phi(y,\alpha)$ *such that for each* $\alpha \in V_0$, $y = (y_1, \ldots, y_n) \in W$ *we have* $\Phi(y,\alpha) \in U_0$ *and*

$$S(\Phi(y,\alpha),\alpha) = S(r_0,\alpha) + \frac{1}{2}\sum_{j=1}^{p} y_j^2 - \frac{1}{2}\sum_{j=p+1}^{n} y_j^2,$$

where p *is the number of positive eigenvalues of the matrix* $S''_{rr}(r_0,\alpha_0)$.

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We apply the Morse lemma to the function

$$S(u,q,\alpha) := F(u,q,\alpha) - F'_u(u_0,q_0,\alpha)u - F'_q(u_0,q_0,\alpha)q$$

in a small neighbourhood of the nondegenerate singular point (u_0, q_0, α_0).

References

M. Abramowitz, and I. A. Stegun, (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 10th printing*, New York: Dover, (1972).

A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, J. Watrous, *One-dimensional quantum walks*, Proc. of the 33rd Annual ACM Symposium on Theory of Computing, (2001).

A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, *Uniform asymptotic solution in the form of an Airy function for semiclassical bound states in one-dimensional and radially symmetric problems*, TMF, 201:3 (2019), 382–414; Theoret. and Math. Phys., 201:3 (2019), 1742–1770.

M. Fedoryuk, *The saddle-point method*, Nauka, Moscow, (1977).

R.P. Feynman, A.R. Hibbs, *Quantum mechanics and path integrals*, New York, McGraw-Hill, (1965).

M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, K. Suzuki, *Dispersive estimates for quantum walks on 1D lattice*, J. Math. Soc. Japan 74:1 (2022), 217–246.

M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, K. Suzuki, *Scattering and inverse scattering for nonlinear quantum walks*, Discrete and Continuous Dynamical Systems, 38:7 (2018), 3687-3703.

N. Pavlova, A. Remizov, *Introduction to singularity theory*, MIPT, (2022).

M. Skopenkov, A. Ustinov, *Feynman checkers: lattice quantum field theory with real time*, submitted (2022), arXiv:2208.14247.

M. Skopenkov, A. Ustinov, *Feynman checkers: towards algorithmic quantum theory*, Russian Math. Surveys 77:3(465) (2022), 73-160, https://arxiv.org/abs/2007.12879.

T. Sunada, T. Tate, *Asymptotic behavior of quantum walks on the line*, J. Funct. Anal. 262 (2012), 2608–2645.

Thank you!