The independence numbers and the chromatic numbers of random subgraphs of Kneser's graphs and their generalizations

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The Erdős–Ko–Rado Theorem

Erdős–Ko–Rado, 1961

Let $[n] = \{1, 2, \dots, n\}$. Assume that $\mathcal{F} \subset \binom{[n]}{r}$ with $r \leqslant n/2$ is such a collection of r-subsets that any two of them intersect. Then $|\mathcal{F}| \leqslant \binom{n-1}{r-1}$.

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Let $[n] = \{1, 2, \dots, n\}$. Assume that $\mathcal{F} \subset \binom{[n]}{r}$ with $r \leqslant n/2$ is such a collection of r-subsets that any two of them intersect and $\mathcal{F}$ is not a star. Then $|\mathcal{F}| \leqslant \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.

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Other stability results were proposed by Balogh, Bohman, Mubayi et al. using the notion of a random hypergraph.

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Lovász, 1978

If $r \leqslant n/2$, then $\chi(KG_{n,r}) = n - 2r + 2$.

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Fix a real number $\varepsilon > 0$ and let $r = r(n)$ be a natural number such that $2 \leqslant r(n) = o(n^{1/3})$. Let $p_c(n,r) = ((r+1)\log n - r\log r)/\binom{n-1}{r-1}$. Then as $n \to \infty$,

$$\mathbb{P}\left(\alpha(KG_{n,r,p}) = \binom{n-1}{r-1}\right) \to \begin{cases} 1 & \text{if } p \geqslant (1+\varepsilon)p_c(n,r) \\ 0 & \text{if } p \leqslant (1-\varepsilon)p_c(n,r). \end{cases}$$

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Successively improved by Das and Tran.

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For example, if $g(n)$ is any growing function and r is arbitrary in the range between 2 and $\frac{n}{2} - g(n)$, then for any fixed p,

$$\chi(KG_{n,r,p}) \sim \chi(KG_{n,r}).$$

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Frankl, Füredi, 1985

For any fixed r, s, there exist $c(r, s), d(r, s)$ such that

$$c(r, s)n^{\max\{s, r-s-1\}} \leqslant \alpha(G(n, r, s)) \leqslant d(r, s)n^{\max\{s, r-s-1\}}.$$

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For any fixed r, s such that $r > 2s + 1$,

$$\alpha(G(n, r, s)) = \binom{n-s-1}{r-s-1} = \Theta\left(n^{r-s-1}\right).$$

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By the way, this agrees perfectly with the results concerning $G(n,p) = G_p(n,1,0)$.

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Local conclusion

If $r \leqslant 2s+1$, then the independence number of the random graph $G_{1/2}(n,r,s)$ behaves like the independence number of the Erdős–Rényi random graph: w.h.p. it increases log times when compared to the initial idependence number. Otherwise, it is stable like its analog for Kneser's graph.

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On the other hand, there are *two completely different* constructions of independent sets with cardinality of order n^2.

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It is very important two emphasize here that the exact value of $\alpha(G(n,r,1))$ is unknown for all values of r!

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Theorem (Nagy, 1972).

If $n \equiv 0 \pmod 4$, then $\alpha(G(n,3,1)) = n$. If $n \equiv 1 \pmod 4$, then $\alpha(G(n,3,1)) = n-1$. If $n \equiv 2,3 \pmod 4$, then $\alpha(G(n,3,1)) = n-2$.

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Theorem (Balogh, Kostochka, A.M., 2012).

If $n = 2^k$, then $\chi(G(n,3,1)) = (n-1)(n-2)/6$.

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Theorem (Pyaderkin, A.M., 2016).

W.h.p.

$$\alpha(G_{1/2}(n,3,1)) \sim 2n \log_2 n.$$