PID Passivity-based Control: Application to Energy and Mechanical Systems

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Layout

Preliminaries

Some Facts and Issues

- PID controllers overwhelmingly dominate engineering applications [in regulation tasks].
- Tuning of gains a difficult task for wide ranging operating systems, where the validity of a linearized approximation is limited.
- Gain scheduling, auto tuning and adaptation help but are time consuming and fragile.
- PID's are passive, hence if the plant is passive, closed–loop is $\mathcal{L}_2$-stable for all gains $\Rightarrow$ tuning is trivialised.
- Under additional assumptions $y(t) \to 0$.
- Key issues:
  - How to identify passive outputs?
  - What if the output reference value is non-zero?
  - Can we go beyond $\mathcal{L}_2$-stability and $y(t) \to 0$?
  - Lyapunov stability [of equilibria]?

Standard PID-PBC

Passive Systems: Hill-Moylan's Theorem

The system

$$\Sigma_{(u,y_{\mathrm{HM}})} \begin{cases} \dot{x} = f(x) + g(x)u \\ y_{\mathrm{HM}} = h(x) + j(x)u \end{cases}$$

with $x \in \mathbb{R}^n$, $u, y_{\mathrm{HM}} \in \mathbb{R}^m$ is cyclo-passive with storage function $H(x)$ if and only if, for some $q \in \mathbb{N}$, there exist mappings $\ell(x) \in \mathbb{R}^q$ and $w(x) \in \mathbb{R}^{q\times m}$ such that

$$\begin{aligned} \nabla^\top H(x) f(x) &= -|\ell(x)|^2 \\ h(x) &= g^T(x)\nabla H(x) + 2w^\top(x)\ell(x) \\ w^\top(x)w(x) &= \frac{1}{2}(j^\top(x) + j(x)) \end{aligned}$$

with $\nabla(\cdot) := (\frac{\partial}{\partial x}(\cdot))^\top$. In that case

$$\dot{H} = y_{\mathrm{HM}}^\top u - |\ell(x) + w(x)u|^2.$$

Remark In the sequel assume $H(x) \geq c \;\Rightarrow\;$ passivity.

Basic PI PBC for Output Regulation [to Zero]

Consider system $\Sigma_{(u, y_{\mathtt{HM}})}$ in closed-loop with the PI PBC

$$\begin{aligned} \dot{x}_c &= y_{\mathtt{HM}} \\ u &= -K_P y_{\mathtt{HM}} - K_I x_c + v, \quad K_P, K_I > 0. \end{aligned}$$

Assume

$$\det[I_m + K_P j(x)] \neq 0 \;\Rightarrow\; \text{Well-posedness.}$$

- The operator $v \mapsto y$ is $\mathcal{L}_2$-stable. More precisely, $\exists \beta \in \mathbb{R}$ such that

$$\int_0^t |y_{\mathtt{HM}}(s)|^2 ds \leq \frac{1}{\lambda_{\min}(K_P)} \int_0^t |v(s)|^2 ds + \beta, \; \forall t \geq 0.$$

- If $v = 0$ and $H(x)$ is proper then $y_{\mathtt{HM}}(t) \to 0$.

Basic PID PBC [for Relative Degree One Systems]

Consider system $\Sigma_{(u,y_{\mathtt{HM}})}$ and $j(x)=0$, with the PID PBC

$$\begin{aligned} \dot{x}_c &= y_{\mathtt{HM}} \\ u &= -K_P y_{\mathtt{HM}} - K_I x_c - K_D \frac{dy_{\mathtt{HM}}}{dt} + v. \end{aligned}$$

with $K_P, K_I, K_D > 0$ and

$$\det[I_m + K_D \nabla^\top h(x) g(x)] \neq 0 \;\Rightarrow\; \text{Well-posedness.}$$

- The operator $v \mapsto y$ is $\mathcal{L}_2$-stable. More precisely, $\exists \beta \in \mathbb{R}$ such that

$$\int_0^t |y_{\mathtt{HM}}(s)|^2 ds \leq \frac{1}{\lambda_{\min}(K_P)} \int_0^t |v(s)|^2 ds + \beta, \; \forall t \geq 0.$$

- If $v=0$ and $H(x)$ is proper then $y_{\mathtt{HM}}(t) \to 0$.

Port–Hamiltonian (pH) Systems

Model and Properties

- PH model of a physical system [with natural output]

$$\Sigma_{(u,y)} : \begin{cases} \dot{x} &= [\mathcal{J}(x) - \mathcal{R}(x)]\nabla H + g(x)u \\ y &= g^\top(x)\nabla H \end{cases}$$

  - $u^\top y$ is power (voltage–current, speed–force, angle–torque, etc.)
  - $\mathcal{J} = -\mathcal{J}^\top$ is the interconnection matrix, specifies the internal power–conserving structure
  - $\mathcal{R} = \mathcal{R}^\top \geq 0$ damping matrix (friction, resistors, etc.)
- PH systems are cyclo–passive $\dot{H} = -\nabla H^\top \mathcal{R} \nabla H + u^\top y$.
- Invariance of pH structure Power preserving interconnection of pH systems is pH.
- Nice geometric structure formalized with notion of Dirac structures.
- Most nonlinear cyclo–passive systems can be written as pH systems. Actually, in (network) modeling is the other way around!

Examples: Nonlinear RLC Circuits

- For any (possibly nonlinear) LC circuit we have

$$\dot{x} = \begin{bmatrix} 0 & \Gamma \\ -\Gamma^\top & 0 \end{bmatrix} \nabla H + gu, \quad y = g^\top \nabla H$$

where $x = \mathrm{col}(q_C, \phi_L)$, $H = H_E(q_C) + H_M(\phi_L)$ – electric plus magnetic energies, Γ comes from Kirchhoff's laws and u are (external) voltage and current sources.

- Example: LTI Series RLC circuit
  - Total energy,

$$H(x) = \frac{1}{2C}x_1^2 + \frac{1}{2L}x_2^2$$

  - Co–energy variables $\nabla H = \mathrm{col}(v_C, i_L)$,
  - PH model, u voltage source

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -R \end{bmatrix}}_{\mathcal{J}-\mathcal{R}} \underbrace{\begin{bmatrix} \frac{x_1}{C} \\ \frac{x_2}{L} \end{bmatrix}}_{\nabla H} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g} u, , \quad y = \nabla_{x_2} H = \frac{x_2}{L} = i_L$$

Mechanical Systems

- State $x = \mathrm{col}(q, p)$, $p := M(q)\dot{q}$ momenta.
- Total energy:

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + U(q)$$

- Assuming linear friction,

$$F = R\dot{q}, \quad R = R^\top \geq 0$$

- PH model, u forces/torques

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & I \\ -I & -R \end{bmatrix} \nabla H + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u \\ y &= \nabla_p H = M^{-1} p \quad (= \dot{q}\ ) \end{aligned}$$

G input matrix (actuated coordinates).

Electromechanical Systems

- Assuming linear magnetics, i.e., $\phi = L(\theta) i \in \mathbb{R}^n$, $L(\theta) = L^\top(\theta) \geq 0$, one mechanical d.o.f., $\theta \in \mathbb{R}$, voltages $u \in \mathbb{R}^m$.
- State $x = \mathrm{col}(\phi, \theta, p)$, $p = m\dot{\theta}$.
- Total energy:

$$H(x) = \frac{1}{2}\phi^\top L^{-1}(\theta)\phi + \frac{1}{2m}p^2 + U(\theta)$$

- Co–energy variables $\nabla H = \mathrm{col}(i, -\tau, \dot{\theta})$, where τ force (torque) of electrical origin.
- PH model

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -R & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \nabla H + \begin{bmatrix} M & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ -\tau_L \end{bmatrix} \\ y &= \mathrm{col}(Mi, \omega), \end{aligned}$$

$\tau_L \in \mathbb{R}$ load torque, $M \in \mathbb{R}^{n \times m}$ defines actuated coordinates.

Power Converters

- More general class of PH models:

$$\dot{x} = [\mathcal{J}(x,u) - \mathcal{R}(x)]\nabla H + g(x,u)$$

- The control u modifies the interconnection and input matrices
- Assuming: fast switching, u is the duty cycle.
- State $x = \text{col}(\phi_L, q_C)$
- For linear L_i, C_i the total energy is

$$H(x) = \frac{1}{2}x_1^\top L^{-1}x_1 + \frac{1}{2}x_2^\top C^{-1}x_2,$$

where $L = \text{diag}\{L_i\}$, $C = \text{diag}\{C_i\}$.

Passive Outputs for Port-Hamiltonian Systems

- Hard to identify $y_{\mathtt{HM}}$ for general (f, g, h, j) systems.
- Clearer picture for pH systems

$$\Sigma_{(u,y)} \begin{cases} \dot{x} &= F(x)\nabla H(x) + g(x)u \\ y &= g^\top(x)\nabla H(x), \end{cases}$$

with

$$F(x) := \mathcal{J}(x) - \mathcal{R}(x) \;\Rightarrow\; F(x) + F^\top(x) \leq 0,$$

- All pH systems are passive but converse not true.
- Key questions
  - Can we generate other passive outputs?
  - With other storage functions?

Power Shaping Passive Output

- Assume $F(x)$ is full rank. The pH system

$$\Sigma_{(u,y_{\rm PS})}\left\{\begin{array}{rcl}\dot{x} & = & F(x)\nabla H(x)+g(x)u \\ y_{\rm PS} & = & -g^\top(x)F^{-\top}(x)[F(x)\nabla H(x)+g(x)u],\end{array}\right.$$

satisfies

$$\dot{H}\leq u^\top y_{\rm PS} \;\Rightarrow\; u\mapsto y_{\rm PS} \text{ is passive.}$$

- Proof: $\underbrace{\dot{x}^\top F^{-1}(x)\dot{x}}_{\leq 0}=\underbrace{\dot{x}^\top\nabla H(x)}_{\dot{H}}+\underbrace{\dot{x}^\top F^{-1}(x)g(x)}_{-y_{\rm PS}}u.$
- Full rank condition can be relaxed using pseudo-inverses.
- Can be extended to

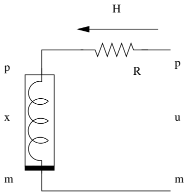
$$y_{\rm EPS}=-g^\top(x)F_d^{-\top}(x)[F(x)\nabla H(x)+g(x)u],$$

for all $F_d(x)$ verifying $F_d(x)+F_d^\top(x)\leq 0$ and

$$\nabla\left(F_d^{-1}F\nabla H\right)=\left[\nabla\left(F_d^{-1}F\nabla H\right)\right]^\top.$$

Physical Interpretation of y_{PS}

Nonlinear RL circuit with x the inductor flux

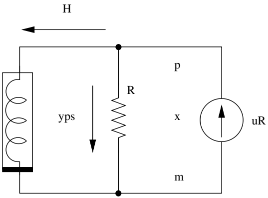


$H(x)$ magnetic energy stored in the inductor. A pH model is

$$\Sigma_{(u,y)} : \begin{cases} \dot{x} & = & -RH'(x) + u \\ y & = & H'(x). \end{cases}$$

Thus, $\dot{H} \leq uy$ with y port current.

Applying Thevenin–Norton transformation



yields the new pH model

$$\Sigma_{(u,y_{\mathrm{PS}})} : \begin{cases} \dot{x} & = & -RH'(x) + u \\ y_{\mathrm{PS}} & = & -H'(x) + \frac{1}{R}u. \end{cases}$$

Hence, $\dot{H} \leq uy_{\mathrm{PS}}$ with y_{PS} current in resistor.

Interpretation in Electro–Mechanical Systems

- The new passive output is a corollary of Thevenin-Norton equivalence.
- $x = \mathrm{col}(\lambda, \theta, p) \in \mathbb{R}^{n_e+2}$, $\lambda \in \mathbb{R}^{n_e}$ magnetic fluxes, $\theta, p \in \mathbb{R}$ mechanical displacement and momenta, u external voltages.
- Electrical equations of this system are of the form

$$\dot{\lambda} = -R_e i + Bu,$$

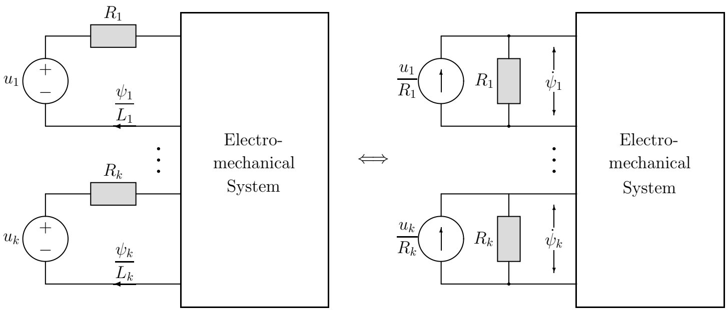
$R_e = R_e^\top > 0 \in \mathbb{R}^{n_e \times n_e}$ resistors, $i \in \mathbb{R}^{n_e}$ currents on the inductors, $\lambda = L(\theta) i$, with $L(\theta) = L^\top(\theta) > 0$ the inductance matrix.

- The natural power port variables u and $y = B^\top L^{-1}(\theta)\lambda$ currents in inductors. Now,

$$u^\top y_{\mathrm{PS}} = u^\top B^\top R_e^{-1} \dot{\lambda},$$

where $R_e^{-1} Bu$ are the current sources obtained from the Norton equivalent of the Thevenin representation, with $\dot{\lambda}$ the associated inductor voltages.

Thevenin-Norton Equivalence



Passive Output of Venkatraman and van der Schaft

The pH system

$$\Sigma_{(u,y_{\rm VV})}\left\{\begin{array}{rcl}\dot{x} & = & F(x)\nabla H(x)+g(x)u\\ y_{\rm VV} & = & [g(x)+2T(x)]^\top\nabla H(x)+[D(x)+S(x)]u,\end{array}\right.$$

where $S(x)\in\mathbb{R}^{m\times m}$, $D(x)\in\mathbb{R}^{m\times m}$, with

$$S(x)=S^\top(x),\ D(x)=-D^\top(x)$$

and $T(x)\in\mathbb{R}^{n\times m}$ verifies

$$\dot{H}=-\left[\begin{array}{cc}\nabla^\top H(x) & u^\top\end{array}\right]\underbrace{\left[\begin{array}{cc}\mathcal{R}(x) & T(x)\\ T^\top(x) & S(x)\end{array}\right]}_{\mathcal{Z}(x)}\left[\begin{array}{c}\nabla H(x)\\ u\end{array}\right]+u^\top y_{\rm VV}.$$

Hence

$$\mathcal{Z}(x)\geq 0\ \Rightarrow\ u\mapsto y_{\rm VV}\ \text{is passive}.$$

A Parameterisation of ALL Passive Outputs

- Introduce the factorisation (always exists)

$$\mathcal{R}(x) = \phi^\top(x)\phi(x),$$

where $\phi(x) \in \mathbb{R}^{q\times n}$, with $q \geq \text{rank}\,\{\mathcal{R}(x)\}$ and define

$$y_{\mathtt{wD}} := h(x) + j(x)u.$$

- The following statements are equivalent.

(S1) The mapping $u \mapsto y_{\mathtt{wD}}$ is passive with storage function $H(x)$.

(S2) For any factorization of the dissipation matrix $\mathcal{R}(x)$ the mappings $h(x)$ and $j(x)$ can be expressed as

$$\begin{aligned} h(x) &= [g(x) + 2\phi^\top(x)w(x)]^\top \nabla H(x) \\ j(x) &= w^\top(x)w(x) + D(x), \end{aligned}$$

for some mappings $w : \mathbb{R}^n \to \mathbb{R}^{q\times m}$ and $D : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, with $D(x)$ skew–symmetric.

PID-PBC Using the Incremental Model

What if the reference value is non-zero?

- Shift the output, $\tilde{y} := y - y_*$ and redefine the PI

$$\dot{x}_c = \tilde{y},\ u = -K_P\tilde{y} - K_I x_c + v.$$

- Is the map $u \mapsto \tilde{y}$ passive? Not, in general (in LTI yes)!
- y_* should be associated to a steady-state operation, i.e., and equilibrium $x_* \in \mathbb{R}^n$
- More precisely, for some $u_* \in \mathbb{R}^m$, we have

$$\begin{aligned} 0 &= f(x_*) + g(x_*)u_* \\ y_* &= h(x_*) + j(x_*)u_*, \end{aligned}$$

- This is true if and only if

$$\begin{aligned} x_* &\in \mathcal{E} := \{x \in \mathbb{R}^n \mid g^\perp(x) f(x) = 0\} \\ u_* &= -g^\dagger(x_*) f(x_*). \end{aligned}$$

$g^\perp(x)$ a full-rank left-annihilator and $g^\dagger(x)$ a pseudo-inverse.

Equilibrium Assignment

The system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x) + j(x)u$$

in closed-loop with

$$\begin{aligned} \dot{x}_c &= \tilde{y} \\ u &= -K_P\tilde{y} - K_I x_c \end{aligned}$$

with $x_* \in \mathcal{E}$, has an equilibrium at

$$(x, x_c) = (x_*, -K_I^{-1}u_*).$$

Passivity of the Incremental pH Model

Consider the pH system

$$\dot{x} = F\nabla H(x) + gu, \;\; y = h(x) + ju$$

- F, g and j constant.
- $u \mapsto y$ passive.
- $H(x)$ convex.

The incremental pH system

$$\dot{x} = F\nabla H(x) + gu_* + g\tilde{u}$$
$$\tilde{y} = h(x) - h(x_*) + j\tilde{u},$$

is passive $\tilde{u} \mapsto \tilde{y}$ with storage function (“Bregman divergence")

$$H_0(x) := H(x) - x^\top \nabla H(x_*).$$

Extensions and Lyapunov Stability

- Can be extended to general (f, g, h, j)-systems verifying

$$[f(x) - f(x_*)]^\top [\nabla H(x) - \nabla H(x_*)] \leq 0.$$

- $H(x)$ strictly convex $\Rightarrow$ $H_0(x)$ has a unique global minimum in x_* and is proper $\Rightarrow$ is a candidate Lyapunov function.
- If so, the PI-PBC

$$\begin{aligned} \dot{x}_c &= \tilde{y} \\ u &= -K_P \tilde{y} - K_I x_c \quad (\Leftrightarrow\ \tilde{u} = -K_P \tilde{y} - K_I \tilde{x}_c) \end{aligned}$$

ensure GS of x_* and GAS if $\tilde{y}$ is detectable.

- No need to know u_* using

$$V_0(x, x_c) := H_0(x) + \frac{1}{2}\|x_c - K_I^{-1} u_*\|^2_{K_I} \ \Rightarrow\ \dot{V} \leq -\|\tilde{y}\|^2_{K_P},$$

with $\|x\|_A := x^\top A x$.

Stabilization of Nonlinear RLC Circuits

System Description

- RLC circuits consisting of interconnections of (possibly nonlinear) lumped dynamic (n_L inductors, n_C capacitors) and static (n_R resistors, n_{v_S} voltage sources and n_{i_S} current sources) elements.
- Capacitors and inductors are defined by

$$i_C = \dot{q}_C, \quad v_C = \nabla H_C(q_C),\; v_L = \dot{\phi}_L, \quad i_L = \nabla H_L(\phi_L),$$

- Total energy

$$H(\phi_L, q_C) := H_L(\phi_L) + H_C(q_C).$$

- For simplicity all current (resp. voltage) controlled resistors are in series with inductors (resp. in parallel with capacitors). Thus,

$$v_{R_{L_i}} = \hat{v}_{R_{L_i}}(i_{L_i}),\; i_{R_{C_i}} = \hat{i}_{R_{C_i}}(v_{C_i})$$

pH Model

- pH model

$$\begin{bmatrix} \dot{\phi}_L \\ \dot{q}_C \end{bmatrix} = \mathcal{J}\nabla H(\phi_L, q_C) - \begin{bmatrix} \hat{v}_{R_L}(\nabla H_L(\phi_L)) \\ \hat{i}_{R_C}(\nabla H_C(q_C)) \end{bmatrix} + gu$$

$$\mathcal{J} = \begin{bmatrix} 0 & -\Gamma \\ \Gamma^\top & 0 \end{bmatrix}, \; g = \begin{bmatrix} -B_{v_S} & 0 \\ 0 & B_{i_S} \end{bmatrix}, \; u = \begin{bmatrix} v_{v_S} \\ i_{i_S} \end{bmatrix},$$

and $\Gamma \in \mathbb{R}^{n_L \times n_C}$, is determined by the circuit topology.

- Port variables

$$y = g^\top \nabla H(\phi_L, q_C) = \begin{bmatrix} -B_{v_S}^\top \nabla H_L(\phi_L) \\ B_{i_S}^\top \nabla H_C(q_C) \end{bmatrix}.$$

Main Result

Consider the nonlinear RLC circuit with $(\phi_L^\star, q_C^\star) \in \mathcal{E}$ and

• Inductors and capacitors are passive and their energy functions are twice continuously differentiable and strictly convex.

• The resistors are passive and their characteristic functions are monotone non–decreasing.

Then, the circuit in closed–loop with the PI-PBC ensures all state trajectories are bounded and

$$\lim_{t\to\infty} \tilde{y}(t) = 0.$$

If, in addition, $\tilde{y}$ is detectable

$$\lim_{t\to\infty} \begin{bmatrix} \tilde{\phi}_L(t) \\ \tilde{q}_C(t) \\ \tilde{x}_C(t) \end{bmatrix} = 0.$$

Regulation and Trajectory Tracking for Bilinear Systems

The Class of Systems

- Model:

$$\dot{x}(t) = Ax(t) + d(t) + \sum_{i=1}^{m} u_i(t) B_i x(t)$$

where $d(t)$ is a known signal.

- There exists $P = P^\top > 0$ such that

$$\begin{aligned} \text{sym}(PA) &=: -Q \leq 0 \\ \text{sym}(PB_i) &= 0, \end{aligned}$$

- Assignable trajectories:

$$\dot{x}_\star(t) = Ax_\star(t) + d(t) + \sum_{i=1}^{m} u_{i\star}(t) B_i x_\star(t)$$

- Error system

$$\dot{\tilde{x}} = (A + \sum_{i=1}^{m} u_i B_i)\tilde{x} + \sum_{i=1}^{m} \tilde{u}_i B_i x_\star.$$

Passivity of the Incremental Model

Define the output $y := \mathcal{C}(x_\star)x$ where

$$\mathcal{C} := \begin{bmatrix} x_\star^\top B_1^\top \\ \vdots \\ x_\star^\top B_m^\top \end{bmatrix} P.$$

The operator $\tilde{u} \mapsto y$ defines a passive map with the storage function

$$V(\tilde{x}) := \frac{1}{2}\tilde{x}^\top P \tilde{x}$$

More precisely

$$\dot{V} = -\tilde{x}^\top Q \tilde{x} + \underbrace{\sum_{i=1}^{m} \tilde{u}_i \tilde{x}^\top P B_i x_\star}_{y^\top \tilde{u}}$$

The PI Tracking Controller

The system in closed-loop with the PI-PBC

$$\dot{x}_C = -\, y$$
$$u = -\, K_P y + K_I x_C + u_\star$$

ensures that trajectories are bounded and $\lim_{t \to \infty} y_a = 0$, where

$$y_a = \begin{bmatrix} \mathcal{C} \\ \mathcal{Q} \end{bmatrix} \tilde{x}.$$

Furthermore, if

$$\text{rank} \begin{bmatrix} \mathcal{C} \\ \mathcal{Q} \end{bmatrix} = n$$

global tracking is achieved.

Application to Power Converters

Model and Passivity Property

- pH Model

$$\dot{x} = \left(J_0 + \sum_{i=1}^{m} J_i u_i - R \right) \nabla H(x) + \left(G_0 + \sum_{i=1}^{m} G_i u_i \right) E$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ duty ratio of the switches and $E \in \mathbb{R}^n$ external sources, with $\sum_{i=1}^{m} G_i u_i E$ switching sources.

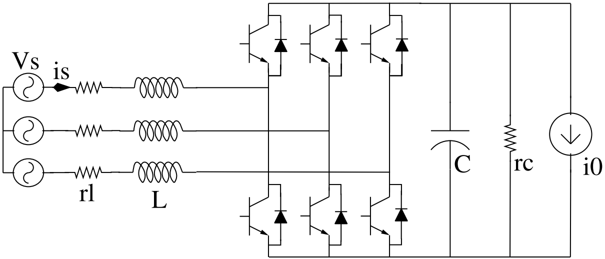
- Total energy stored in inductors and capacitors: $H(x) = \frac{1}{2} x^\top Q x$.
- Passivity of the incremental model. Define $y := \mathcal{C} x$, where

$$\mathcal{C} := \begin{bmatrix} E^\top G_1^\top - (x^\star)^\top Q J_1 \\ \vdots \\ E^\top G_m^\top - (x^\star)^\top Q J_m \end{bmatrix} Q \in \mathbb{R}^{m \times n}.$$

The map $\tilde{u} \mapsto \tilde{y}$ is passive with storage function $V(\tilde{x}) = \frac{1}{2} \tilde{x}^\top Q \tilde{x}$. More precisely,

$$\dot{V} = -\tilde{x}^\top Q R Q \tilde{x} + \tilde{y}^\top \tilde{u}$$

I. Three-phase Rectifier



Model in *dq* frame

$$
\begin{aligned}
\dot{\phi}_d &= -\frac{r_L}{L}\phi_d + \omega\phi_q - \frac{\mu_0}{C}u_1 q_C + V \\
\dot{\phi}_q &= -\frac{r_L}{L}\phi_q - \omega\phi_d - \frac{\mu_0}{C}u_2 q_C \\
\dot{q}_C &= \frac{\mu_0}{L}u_1\phi_d + \frac{\mu_0}{L}u_2\phi_q - \frac{1}{Cr_c}q_C - I
\end{aligned}
$$

cont'd

- pH model

$$x = \begin{pmatrix} \phi_d \\ \phi_q \\ q_C \end{pmatrix}, \; G_0 E = \begin{pmatrix} V \\ 0 \\ -I \end{pmatrix}, \; R = \begin{pmatrix} r_L & 0 & 0 \\ 0 & r_L & 0 \\ 0 & 0 & \frac{1}{r_c} \end{pmatrix},$$

$$Q = \begin{pmatrix} \frac{1}{L} & 0 & 0 \\ 0 & \frac{1}{L} & 0 \\ 0 & 0 & \frac{1}{C} \end{pmatrix}, \; J_0 = L\omega \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_1 = \mu_0 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; J_2 = \mu_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

- Control objective $x_3(t) \to x_3^\star > 0$ power factor $\approx 1 \;\Rightarrow\; x_2(t) \to 0$.
- Assignable equilibria

$$x_1^\star = \frac{L}{2r_L}\left(V - \sqrt{V^2 - \frac{4r_L}{C^2 r_c} x_3^{*2} - \frac{4r_L}{C} I x_3^\star}\right).$$

cont'd

- The circuit does not have switched external sources $\Rightarrow\ y^\star = 0$.
- Passive output

$$y = \frac{x_3^\star \mu_0}{LC} \begin{bmatrix} \frac{x_1^\star}{x_3^\star} x_3 - x_1 \\ -x_2 \end{bmatrix}.$$

- The detectability condition is satisfied $\Rightarrow$ PI-PBC ensures GAS.
- Relation with Akagi's PQ method. With reactive power injection, i.e., $x_2^* \neq 0$ and in co-energy variables

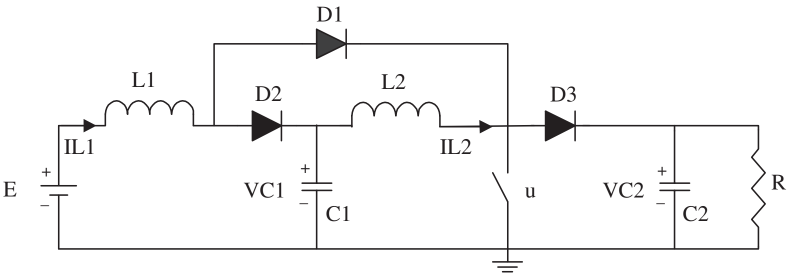
$$y = k \begin{bmatrix} v_C^\star i_d - i_d^\star v_C \\ v_C^* i_q - i_q^* v_C \end{bmatrix},\ k \in \mathbb{R}_+.$$

In Akagi two nested PI's to make AC power $P := v_d i_d$ equal to DC power $P_{\rm DC} := v_C i_{\rm DC}$. Define $P^* := v_d i_d^*$ and $P_{\rm DC}^* := v_C^* i_{\rm DC}$. Then

$$\begin{aligned} P^* P_{\rm DC} &= P_{\rm DC}^* P \Leftrightarrow y_1 = 0 \\ Q^* P_{\rm DC} &= P_{\rm DC}^* Q \Leftrightarrow y_2 = 0, \end{aligned}$$

where $Q := v_d i_q$ is reactive power. Thus, PI-PBC also achieves power equalisation.

II. Quadratic Converter



The goal is $V_{C2}(t) \to V_d$.

Port–Hamiltonian Model

$$\dot{x} = (J_0 + J_1 u - R)\nabla H(x) + B,$$

$$x = \begin{pmatrix} i_{L1} & i_{L2} & v_{C1} & v_{C2} \end{pmatrix},\ B = \begin{pmatrix} \frac{E}{L_1} & 0 & 0 & 0 \end{pmatrix}^\top$$

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r_L C_2^2} \end{pmatrix},\ Q = \begin{pmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_2 \end{pmatrix}$$

$$J_0 = \frac{1}{C_1 L_2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} J_1 = \begin{pmatrix} 0 & 0 & -\frac{1}{L_1 C_1} & 0 \\ 0 & 0 & 0 & -\frac{1}{L_2 C_2} \\ \frac{1}{L_1 C_1} & 0 & 0 & 0 \\ 0 & \frac{1}{L_2 C_2} & 0 & 0 \end{pmatrix}.$$

Incrementally Passive Output

- The admissible equilibria can be parameterized by the reference $x_4^\star$ as follows

$$x^\star := \begin{bmatrix} \frac{1}{r_L(u^\star)^2} & \frac{1}{r_L u^\star} & u^\star & 1 \end{bmatrix}^\top x_4^\star$$

where $u^\star = \sqrt{\frac{E}{x_4^\star}}$ is the corresponding constant control.

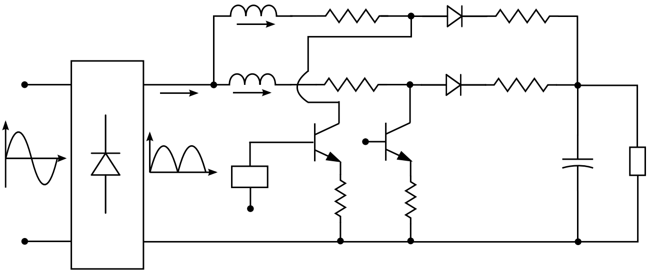
- The output

$$\tilde{y} = -\sqrt{E v_d} x_1 - v_d x_2 + \frac{v_d^2}{E r_L} x_3 + \frac{v_d}{r_L}\sqrt{\frac{v_d}{E}} x_4,$$

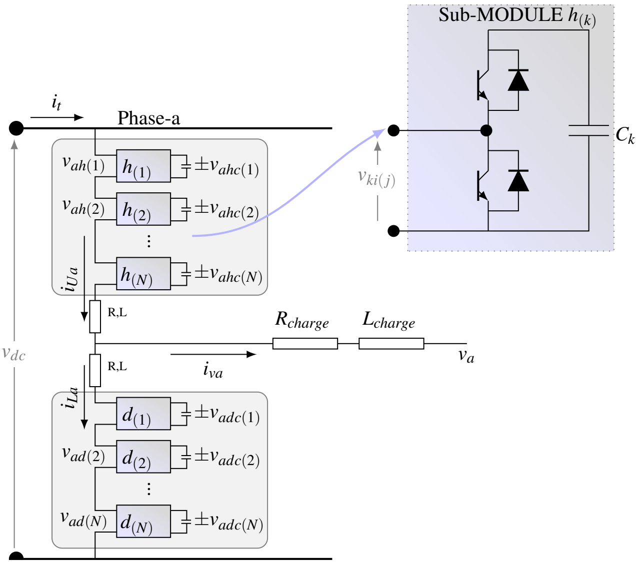
is incrementally passive and detectable

- The equilibrium $x^\star$ can be rendered GAS with the PI-PBC.
- The only parameters that are required are r_L and E, and that the tuning gains can take arbitrary positive values.
- Adaptation added to estimate r_L, preserving the stability properties.

III. Interleave Boost Converter



IV. Modular Multilevel Converter



Lyapunov Stabilisation via PID-PBC

Energy Shaping [$\Rightarrow$ Constructing a Lyapunov Function]

- Define the function

$$U(x, x_c) := H(x) + \frac{1}{2}\|h(x)\|_{K_D}^2 + \frac{1}{2}\|x_c\|_{K_I}^2.$$

- We know that

$$\dot{U} = -\|\nabla H(x)\|_{\mathcal{R}}^2 - \|y_{\mathtt{wD}}\|_{K_P}^2 \leq 0,$$

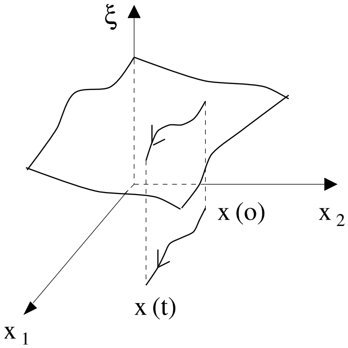
From a La Salle-based analysis $y_{\mathtt{wD}}(t) \to 0$.

- To prove Lyapunov stability we need a Lyapunov function: finding a function $H_d : \mathbb{R}^n \to \mathbb{R}$ such that

$$U(x, x_c) \equiv H_d(x).$$

Since $H_d(x(t))$ is non-decreasing it will be a bona fide Lyapunov function if it is positive definite.

Basic Idea



- Express x_c as function of x $\Leftrightarrow$ find a first integral $\Leftrightarrow$ solving a PDE.
- We look for functions $\gamma : \mathbb{R}^n \to \mathbb{R}^m$ such that the level sets

$$\Omega_\kappa := \{(x, x_c) \mid x_c = \gamma(x) + \kappa\}$$

are invariant, with κ determined by the ICs.

- That is true if and only if

$$\dot{x}_c = \dot{\gamma} = \nabla^\top \gamma [f(x) + g(x)u]$$

- In that case

$$H_d(x) = U(x, \gamma(x) + \kappa)$$

Energy Shaping via Generation of First Integrals

Consider the pH system $\Sigma_{(u,y_{\mathtt{wD}})}$ with $\dot{x}_c = y_{\mathtt{wD}}$. Assume there exists mappings $w(x)$ and $D(x)$ such that the PDE

$$\begin{bmatrix} [\nabla H(x)]^\top F^\top(x) \\ g^\top(x) \end{bmatrix} \nabla\gamma(x) = \begin{bmatrix} [\nabla H(x)]^\top [g(x) + 2\phi^\top(x) w(x)] \\ w^\top(x) w(x) - D(x) \end{bmatrix}$$

admits a solution $\gamma : \mathbb{R}^n \to \mathbb{R}^m$. Then,

$$x_c = \gamma(x) + \kappa$$

for some $\kappa \in \mathbb{R}$. Consequently,

$$U(x, x_c) = H_d(x) = H(x) + \frac{1}{2}\|h(x)\|^2_{K_D} + \frac{1}{2}\|\gamma(x) + \kappa\|^2_{K_I}.$$

Proof

Established showing that the PDE is equivalent to

$$y_{\mathtt{wD}} = \dot{\gamma}. \qquad (\star)$$

Consequently, using $\dot{x}_c = y_{\mathtt{wD}}$ and integrating we get

$$x_c = \gamma(x) + x_c(0) - \gamma(x(0)).$$

Now, $(\star)$ is equivalent to

$$[g(x) + 2\phi^\top(x)w(x)]^\top \nabla H(x) + [w^\top(x)w(x) + D(x)]u$$
$$= [\nabla\gamma(x)]^\top [F(x)\nabla H(x) + g(x)u].$$

The proof is completed equating the terms dependent and independent on u and factoring $\nabla\gamma(x)$.

Control by Interconnection

- Integral control can be represented as a pH system

$$\begin{aligned} \dot{x}_c &= u_c \\ y_c &= \nabla H_c(x_c) \end{aligned}$$

with state $x_c \in \mathbb{R}^m$, port variables $u_c, y_c \in \mathbb{R}^m$ and Hamiltonian

$$H_c(x_c) := \frac{1}{2}\|x_c\|^2_{K_I}.$$

- Add a power preserving interconnection

$$\begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0_{m\times m} & -I_m \\ I_m & 0_{m\times m} \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix}.$$

- The closed-loop is pH with total energy function $H(x) + H_c(x_c)$.
- In CbI the energy is shaped generating quantities that are conserved by the open loop pH system for all energy functions $H(x)$, called Casimir functions: $\mathcal{C}(x) \in \mathbb{R}^m$.

First Integrals vs Casimir Functions

- If $\dot{x} = \dot{\mathcal{C}}$ the function

$$H_d(x) := H(x) + H_c(\mathcal{C}(x) + \kappa)$$

satisfies $\dot{H}_d \leq 0$. Given $\mathcal{C}(x)$, $H_d(x)$, can be shaped selecting H_c.

- Casimirs are the solutions of the PDE

$$\begin{bmatrix} F^\top(x) \\ g^\top(x) \end{bmatrix} \nabla \mathcal{C}(x) = \begin{bmatrix} g(x) + 2\phi^\top(x) w(x) \\ w^\top(x) w(x) - D(x) \end{bmatrix}.$$

- Comparing with the PDE of PID-PBC no term $\nabla H(x)$. Hence, the set of Casimirs is strictly contained in the set of solutions of our PDE.
- On the other hand, it is possible to give verifiable conditions such that the Casimirs PDE reduces to a simple integration.
- Casimirs solely determined by $F(x)$, hence, physically appealing and with a nice geometric interpretation

Solving the PDE

Consider the pH system $\Sigma_{(u,y_{\mathtt{wD}})}$ verifying

$$\begin{aligned} F^\top(x)[F^\dagger(x)]^\top(x)F(x) &= F(x) \\ \mathrm{span}\{g(x)\} &\subseteq \mathrm{span}\{F(x)\}. \end{aligned}$$

Assume $F^\dagger(x)g_i(x)$, are gradient vector fields, that is,

$$\nabla[F^\dagger(x)g_i(x)] = \left(\nabla[F^\dagger(x)g_i(x)]\right)^\top \left[\Leftrightarrow \ \exists\gamma_i(x) \mid \nabla\gamma_i(x) = -F^\dagger(x)g_i(x)\right].$$

Then,

$$\gamma_i(x) = -\int_0^1 x^\top F^\dagger(sx)g_i(sx)ds,$$

is a solution of the Casimir's PDE with

$$\begin{aligned} w(x) &= \phi(x)F^\dagger(x)g(x) \\ D(x) &= -g^\top(x)[F^\dagger(x)]^\top(x)\mathcal{J}(x)F^\dagger(x)g(x). \end{aligned}$$

Input-Output Change of Coordinates

- Introduce a full rank matrix M and define

$$\bar{u} := M^{-1}(x)u, \ \bar{y} := M^\top(x)y_{\mathtt{wD}}.$$

- Clearly, the power balance inequality is preserved

$$\dot{H} \leq u^\top y = \bar{u}^\top \bar{y}.$$

- Consider the power shaping output, the new output is

$$\bar{y} = -M^\top(x)g^\top(x)F^{-\top}(x)\dot{x}.$$

- There existes a mapping $\gamma(x)$ such that $\bar{y} = \dot{\gamma}$ iff

$$\text{rank}\left\{\left[\Lambda(x) \ \ \vdots \ \ [\Lambda_i(x), \Lambda_j(x)]\right]\right\} = n - m,$$

where $\Lambda \in \mathbb{R}^{n\times(n-m)}$ is full rank and verifies

$$g^\top(x)F^{-\top}(x)\Lambda(x) = 0$$

Static State-Feedback Implementation

Equilibrium Assignment

Consider the pH system $\Sigma_{(u, y_{\mathtt{wD}})}$ with $w(x)$ and $D(x)$ such that the PDE admits a solution $\gamma(x)$. Fix an equilibrium $x^\star \in \mathcal{E}$ and consider the PID-PBC

$$u = -K_P y_{\mathtt{wD}} - K_I(\gamma(x) - \gamma^\star) - K_D \frac{dy_{\mathtt{wD}}}{dt},$$

Then, $x^\star$ is an equilibrium point of the closed-loop system.

Lyapunov Stabilization

Fix $x^\star \in \mathcal{E}$. Consider the pH system $\Sigma_{(u,y_{\mathtt{wD}})}$ with the (static state-feedback) PID-PBC where $w(x)$ and $D(x)$ such that the PDE admits a solution $\gamma(x)$. Define

$$H_d(x) = H(x) + \frac{1}{2}\|h(x)\|^2_{K_D} + \frac{1}{2}\|\gamma(x) - \gamma(x^\star)\|^2_{K_I},$$

and assume

$$x^\star = \arg\min H_d(x).$$

(i) The closed–loop system has a stable equilibrium at $x = x^\star$ with Lyapunov function $H_d(x)$.

(ii) The equilibrium is asymptotically stable if $y_{\mathtt{wD}}$ is a detectable output for the closed–loop system.

(iii) The stability properties are global if $H_d(x)$ is radially unbounded.

Relation with Classical PBCs

- Energy–balancing PBC: $\dot{H}_a = -u_{\mathtt{EB}}^\top y_{\mathtt{PS}}$. Fix $K_P = 0$ then, the PID-PBC is an EB-PBC with added energy function

$$H_a(x) := \frac{1}{2}\|\gamma(x) + C\|^2_{K_I}.$$

- IDA-PBC: Control $u = u_{\mathtt{IDA}}(x)$ such that the closed–loop has the form

$$\dot{x} = F_d(x)\nabla H_{\mathtt{IDA}}(x).$$

Assignable $H_{\mathtt{IDA}}(x)$ characterized by the solutions of the PDE

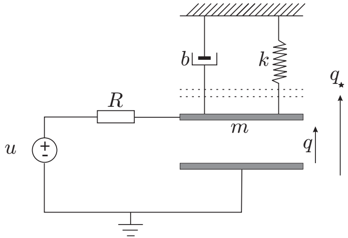
$$g^\perp(x)\left[F_d(x)\nabla H_{\mathtt{IDA}}(x) - F(x)\nabla H(x)\right] = 0,$$

and the control is uniquely defined as

$$u_{\mathtt{IDA}}(x) := g^\dagger(x)\left[F_d(x)\nabla H_{\mathtt{IDA}}(x) - F(x)\nabla H(x)\right].$$

Fix $K_P = 0$ and select $F_d(x) = F(x)$. Then, the energy function $H_d(x)$ and the control of the PID-PBC satisfy the IDA-PBC equations.

Micro Electro–mechanical Optical Switch



- pH model

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b & 0 \\ 0 & 0 & -\frac{1}{r} \end{bmatrix} \nabla H(x) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

- Energy function of the system is

$$H(x) = \frac{1}{2m} x_2^2 + \frac{1}{2} a_1 x_1^2 + \frac{1}{4} a_2 x_1^4 + \frac{1}{2c_1(x_1 + c_0)} x_3^2.$$

cont'd

- Assignable equilibria: $x_1 \in \mathbb{R}_{>0}$,

$$x_{2_\star} = 0$$
$$x_{3_\star} = (c_0 + x_{1_\star})\sqrt{2c_1 x_{1_\star}(a_1 + a_2 x_{1_\star}^2)}$$

and the goal is to stabilize at $x_{1_\star} > 0$.

- F is full rank and $y_{\mathrm{PS}} = \frac{1}{r}\dot{x}_3$, therefore $\gamma(x) = \frac{1}{r}x_3$.
- Finally

$$\nabla^2 H_d(x_\star) = \begin{bmatrix} a_1 + 3a_2 x_{1_\star}^2 + d_1^2 d_2 & 0 & -d_1 d_2 \\ 0 & \frac{1}{m} & 0 \\ -d_1 d_2 & 0 & d_2 \end{bmatrix} + \frac{K_I}{r}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $d_1, d_2 > 0$. Then, $\nabla^2 H_d(x_\star) > 0$ for all $K_I > 0$ $\Rightarrow$ $x_\star$ is a stable equilibrium for the closed–loop system.

- Asymptotic stability also follows.

LTI systems: Controllability is Not Enough

- IDA-PBC for LTI systems is a universal stabiliser, in the sense that it is applicable to all stabilisable systems.
- Stabilisability is not enough for IDA–PBC of mechanical system.
- For the PID–PBC presented here even controlability is not enough.
- For LTI system F and g are constant

$$H(x) = \frac{1}{2} x^\top Q x,$$

and $x_\star = 0$.

- The PID-PBC is $u = Kx$ with

$$K := \left(I - K_P g^\top F^{-\top} g\right)^{-1} \left(K_P g^\top F^{-\top} F Q + K_I g^\top F^{-\top}\right).$$

cont'd

- Consider the controllable LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a_1 & 1 - a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \; a_1 < 0$$

- Admits a pH representation $\dot{x} = FQx + gu$ with $g := \mathrm{col}(0, 1)$,

$$F := \begin{bmatrix} -1 & a_1 \\ \frac{1}{2} a_1 & -a_1^2 \end{bmatrix}, \; Q := -\frac{2}{a_1^2} \begin{bmatrix} a_1^2 & a_1 \\ a_1 & 1 - \frac{a_1}{2} \end{bmatrix},$$

which satisfies $F + F^\top < 0$ and the assumptions.

- The closed–loop is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a_1 - a_1 \tilde{k} & 1 - a_1 - \tilde{k} \end{bmatrix} x$$

where

$$\tilde{k} := \frac{2}{a_1^2} \left(1 + \frac{2K_P}{a_1^2} \right)^{-1} (K_I + K_P).$$

It is unstable for all values of K_P and K_I.

CbI vs PID-PBC and use of General Output

- Consider a pH system with $H(x) = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}x_3^2$ and

$$\mathcal{J} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathcal{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, g(x) = \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix}.$$

- The control objective is to stabilize $x^\star = (0, 0, x_3^\star)$, with $x_3^\star < 0$.

(i) The system is not stabilisable via CbI.

(ii) Nor with PID-PBC with the power shaping output.

(iii) It is stabilisable with the PID-PBC using the output

$$y = (g + 2\phi^\top w)^\top \nabla H + w^\top w u$$

with

$$w = \begin{bmatrix} x_1 \\ 0 \\ -1 \end{bmatrix}, \quad \phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

PID-PBC of Mechanical Systems

Model and Control Objective

- pH model

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u$$

where $H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q)$, $rank(G) = m < n$.

- EL model

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla V(q) = G(q)u.$$

- Desired Lyapunov function: $H_d(q,p) = \frac{1}{2} p^\top M_d^{-1}(q) p + V_d(q)$
  - $M_d(q) = M_d^\top(q) > 0$
  - $q_\star = \arg\min V_d(q)$.
- Objective Assign $H_d(q,p)$ as a Lyapunov function to the closed loop via PID-PBC for a class of mechanical systems.

Class of Systems

Partition $q = \text{col}(q_a, q_u)$, with $q_a \in \mathbb{R}^m$ and $q_u \in \mathbb{R}^{n-m}$ and

$$M(q) = \begin{bmatrix} m_{aa}(q) & m_{au}(q) \\ m_{au}^\top(q) & m_{uu}(q) \end{bmatrix}$$

A0. The distribution spanned by the columns of $G(q)$ is involutive. Equivalently, there exists (state and input) change of coordinates so that $G = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$.

A1. The inertia matrix depends only on q_u, *i.e.*, $M(q) = M(q_u)$.

A2. The sub-block matrix m_{aa} of the inertia matrix is constant.

A3. The potential energy can be written as

$$V(q) = V_a(q_a) + V_u(q_u).$$

Passive Outputs

- Define the signals

$$y_u := -m_{aa}^{-1} m_{au}(q_u)\dot{q}_u, \;\; y_a := m_{aa}^{-1} m_{au}(q_u)\dot{q}_u + \dot{q}_a.$$

- Apply the inner-loop control

$$u = \nabla V_a(q_a) + v$$

- The maps $v \mapsto y_a$ and $v \mapsto y_u$ are passive with storage functions

$$\begin{aligned} H_u(q_u, \dot{q}_u) &:= \frac{1}{2}\dot{q}_u^\top (m_{uu} - m_{au}^\top m_{aa}^{-1} m_{au})\dot{q}_u + V_u(q_u) \\ H_a(q, \dot{q}) &:= \frac{1}{2}\dot{q}^\top \begin{bmatrix} m_{au}^\top m_{aa}^{-1} m_{au} & m_{au}^\top \\ m_{au} & m_{aa} \end{bmatrix} \dot{q}. \end{aligned}$$

More precisely

$$\dot{H}_a = v^\top y_a, \quad \dot{H}_u = v^\top y_u.$$

Remarks on the Assumptions

- Assumption **A1** implies that the shape coordinates coincide with the unactuated coordinates.
- **A1** and **A2** $\Rightarrow \exists T(q_u) \in \mathbb{R}^{n \times n}$ of the form

$$T(q_u) = \begin{bmatrix} T_1(q_u) & 0_{(n-m)\times m} \\ T_2(q_u) & T_3 \end{bmatrix},$$

with $T_3 \in \mathbb{R}^{m \times m}$ constant s.t. $M^{-1}(q_u) = T(q_u)T^\top(q_u)$.

- This class contains many benchmark examples:
  - robots with flexible links (modulo **A3**),
  - cart–pole,
  - pendubot,
  - spherical pendulum on a puck,
  - disk-on-disk.

Well-posedness and Energy Shaping Assumptions

A4. The rows of $m_{au}(q_u)$ are gradient vector fields, that is,

$$\nabla(m_{au})^i = [\nabla(m_{au})^i]^\top, \ \forall i \in \bar{m}.$$

Equivalently, there exists a function $V_N : \mathbb{R}^{n-m} \to \mathbb{R}^m$ such that

$$\dot{V}_N = -m_{au}(q_u)\dot{q}_u.$$

A5. There exist $k_e, k_a, k_u \in \mathbb{R}, K_D, K_I \in \mathbb{R}^{m\times m},\ K_D, K_I \geq 0$, s.t.
(i) The matrix $K : \mathbb{R}^{n-m} \to \mathbb{R}^{m\times m}$

$$K(q_u) := k_e I_m + k_a K_D T_3 T_3^\top + k_u K_D T_2(q_u) T_2^\top(q_u).$$

verifies

$$\det[K(q_u)] \neq 0.$$

cont'd

(ii) The matrix

$$M_d(q_u) = \begin{bmatrix} A(q_u) & k_a k_u T_2^\top(q_u) K_D T_3 \\ k_a k_u T_3^\top K_D T_2(q_u) & D(q_u) \end{bmatrix}^{-1}$$

with

$$\begin{aligned} A(q_u) &:= k_u^2 T_2^\top(q_u) K_D T_2(q_u) + k_e k_u I_s \\ D(q_u) &:= k_e k_a I_m + k_a^2 T_3^\top K_D T_3. \end{aligned}$$

is positive definite.

(iii) The function

$$V_d(q) := k_e k_u V_u(q_u) + \frac{1}{2} \| k_a q_a + (k_u - k_a) V_N(q_u) \|_{K_I}^2,$$

has an isolated minimum in q_*.

Main Result

Fix $q^\star \in \mathbb{R}^n$ s.t. $\nabla V_u(q_u^\star) = 0$. The system in closed-loop with

$$u = \nabla V_a(q_a) + v$$

and the PID-PBC

$$k_e v \quad = \quad -[K_P y_d + K_I(\gamma(q) - \gamma(q^\star)0 + K_D \dot{y}_d]$$

with

$$y_d := k_a y_a + k_u y_u.$$

has a globally stable equilibrium at $(q, \dot{q}) = (q_\star, 0)$ with Lyapunov function

$$H_d(q, \dot{q}) = \frac{1}{2}\dot{q}^\top M_d(q)\dot{q} + V_d(q).$$

Proof

- Note that

$$y_d := k_a y_a + k_u y_u.$$

- Consequently $v \mapsto y_d$ is passive with storage function

$$k_a H_a(q_u, \dot{q}) + k_u H_u(q_u, \dot{q}_u).$$

- Consequently the function

$$U(q, \dot{q}, x_c) := k_e[k_a H_a(q_u, \dot{q}) + k_u H_u(q_u, \dot{q}_u)] + \frac{1}{2}\|x_c\|^2_{K_I} + \frac{1}{2}\|y_d\|^2_{K_D},$$

verifies $\dot{U} \leq -\|y_d\|^2_{K_P}$.

- The proof is completed proving that Assumption **A4** ensures

$$x_c(t) = \int_0^t y_d(s)ds = k_a q_a(t) - (k_a - k_u) V_N(q_u(t)) + \kappa$$

$$\Rightarrow \; H_d(q, \dot{q}) \equiv U(q, \dot{q}, x_c).$$

Tracking Constant Speed Trajectories

Result can be extended verbatim to track ramps in the actuated coordinate.

Example: Tracking for inverted pendulum on a cart

- 2–DOF example $G = \text{col}(0,1)$, q_u is the angle of the pendulum and q_a the position of the cart.
- The model parameters

$$M(q_u) = \begin{bmatrix} 1 & b\cos(q_u) \\ b\cos(q_u) & m_3 \end{bmatrix}, \; V(q_u) = a\cos(q_u).$$

Assumptions **A1–A4** are satisfied.

- Objective to stabilize the up-right vertical position of the pendulum and impose a ramp trajectory to the cart $q_u^* = 0$, $q_a^*(t) = rt$, $r \in \mathbb{R}$.

Verifying Energy Shaping Assumption **A5**

- PID-PBC with $k_a = 1$

$$M_d^{-1}(q_u) = \begin{bmatrix} k_u^2 K_D \frac{b^2\cos^2(q_u)}{m_3\delta(q_u)} + k_e k_u & -k_u K_D b \frac{\cos(q_u)}{m_3\sqrt{\delta(q_u)}} \\ -k_u K_D b \frac{\cos(q_u)}{m_3\sqrt{\delta(q_u)}} & k_e + \frac{K_D}{m_3} \end{bmatrix}$$

with $\delta(q_u) := m_3 - b^2\cos^2(q_u) > 0$ and

$$V_d(q) = a k_e k_u \cos(q_u) + \frac{K_I}{2}\left[q_a + \frac{(1-k_u)}{m_3}\underbrace{b\sin(q_u)}_{V_N(q_u)}\right]^2.$$

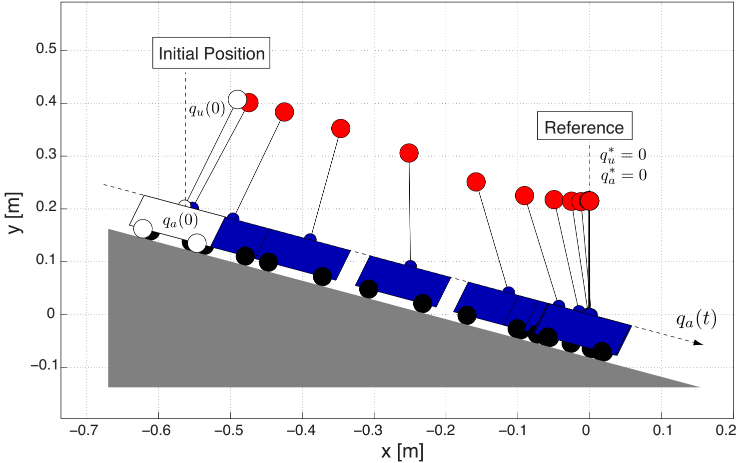
- $0 = \arg\min V_d(q) \;\Leftrightarrow\; k_e k_u < 0$.
- No gains s.t. $M_d(q_u) > 0$ for $|q_u| \geq \frac{\pi}{2} \;\Rightarrow$ stability only local
- Given any $\epsilon > 0$, there exists gains s.t.

$$M_d(q_u) > 0,\; K(q_u) \neq 0, \quad \forall q_u \in \left[\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon\right].$$

Implies the domain of attraction is the whole (open) half plane.

Avoiding Cancellation of $V_a(q_a)$: Example

Potential energy $V(q) = mg\ell\cos(q_u) - (M_c + m)g\sin(\psi)q_a$.



https://youtu.be/CGInoXkR0FA.♡

https://youtu.be/YBcI9WIaQa0