Semidefinite Relaxation and Statistical Estimation

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What the story is about

♣ **Ultimate Goal:** Given noisy observation

$$\omega = Ax + \eta$$

- x – *unknown signal* known to belong to a given *signal set* $\mathcal{X} \subset \mathbb{R}^n$
- A – given $m \times n$ *sensing matrix*
- η – observation noise,

we want to recover linear image Bx of the signal.

- B – given $\nu \times n$ matrix.

♠ **Models of noise:**

- *bounded noise*: all we know is that $\eta \in \mathcal{H} \leftarrow$ given compact set in $\mathbb{R}^m$
- *random noise*: η is random with covariance matrix $\mathrm{Cov}[\eta] := \mathbf{E}\{\eta\eta^T\} \in \Theta \leftarrow$ given compact subset of the cone $\mathbf{S}^m_+$ of positive semidefinite $m \times m$ matrices.

♠ **An estimate** is (any) function $\widehat{x}(\omega) : \mathbb{R}^m \to \mathbb{R}^\nu$. We quantify performance of an estimate by its *risk*:

bounded noise:

$$\mathrm{Risk}_{\|\cdot\|,\mathcal{H}}[\widehat{x}|\mathcal{X}] = \sup_{\substack{x\in\mathcal{X}\\ \eta\in\mathcal{H}}} \|\widehat{x}(Ax+\eta) - Bx\|$$

random noise:

$$\mathrm{Risk}_{\|\cdot\|,\Theta}[\widehat{x}|\mathcal{X}] = \sup_{\substack{x\in\mathcal{X}\\ \eta:\mathrm{Cov}[\eta]\in\Theta}} \mathbf{E}\left\{\|\widehat{x}(Ax+\eta) - Bx\|\right\}$$

- $\|\cdot\|$ – given norm on $\mathbb{R}^\nu$

$$\omega = Ax + \eta \quad ?? \Rightarrow ?? \quad \widehat{x}(\omega) \approx Bx$$

♣ We are about to demonstrate that

• *Under appropriate assumptions on $\mathcal{X}, \|\cdot\|, \mathcal{H}$ one can build, in a computationally efficient fashion, a "presumably good" linear estimate*

$$\widehat{x}_H(\omega) = H^T\omega$$

• *The resulting estimate is nearly optimal*, in certain precise sense, *among all estimates, linear and nonlinear alike.*

Note: Achieving these goals must impose some restrictions on the "geometry" of the data $\mathcal{X}, \|\cdot\|, \mathcal{H}, \Theta$. In what follows we assume that

- Θ, if relevant, is a convex compact subset of the *interior* of $\mathbf{S}^m_+$
- $\mathcal{X}$ and the unit ball $\mathcal{B}_* = \{u : \|u\|_* \leq 1\}$ of the norm *conjugate* to $\|\cdot\|$:
$$\|u\|_* = \max\{u^Tv : \|v\| \leq 1\},$$
same as $\mathcal{H}$, if relevant, are *ellitopes* or *spectratopes*.

Why linear estimates?

♠ As it was announced, *a "nearly optimal" linear estimate can be built in a computationally efficient fashion.*

♠ **In contrast,**

• *Exactly minimax optimal* estimate is *unknown* even in the simplest case when the observation is

$$\omega = x + \eta$$

with $\eta \sim \mathcal{N}(0, \sigma^2)$ and $x \in \mathcal{X} = [-1, 1]$

• *"Standard" Maximum Likelihood estimate* can be disastrously bad even in the simple case

$$\omega = x + \eta,$$
$$\eta \sim \mathcal{N}(0, \sigma^2 I_n),\ \mathcal{X} = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\},\ Bx = x_1$$

In this case, natural implementation of ML estimate is

Build signal $\widetilde{x}$ most likely yielding the observation:

$$\omega \mapsto \widetilde{x} = \underset{\|u\|_2 \leq 1}{\operatorname{argmin}} \|\omega - u\|_2$$

and take $\widetilde{x}_1$ as the estimate of $Bx = x_1$.

For σ small and fixed and n large, with overwhelming probability $\widetilde{x} = \omega / \|\omega\|_2 \approx \omega / \sqrt{n\sigma^2}$, implying that $|\widetilde{x}_1| \leq \frac{O(1)}{\sigma\sqrt{n}}$, and the risk of the ML estimate is $O(1)$, as compared to the minimax optimal risk $O(\sigma)$.

Ellitopes and Spectratopes

♠ **Basic ellitope** in $\mathbb{R}^N$ is a *bounded* set $\mathcal{Z}$ given by representation

$$\mathcal{Z} = \{z \in \mathbb{R}^N : \exists t \in \mathcal{T} : z^T S_k z \leq t_k, 1 \leq k \leq K\}$$

where

- $S_k \succeq 0$, $k \leq K$
- $\mathcal{T} \subset \mathbb{R}^K_+$ is convex compact set which contains a positive vector and is *monotone*: $0 \leq t' \leq t \in \mathcal{T}$ implies that $t' \in \mathcal{T}$.

♠ **Examples:**
A. *Bounded* intersection of K ellipsoids/elliptic cylinders centered at the origin ($\mathcal{T} = [0,1]^K$)
B. $\|\cdot\|_p$-norm ball, $2 \leq p \leq \infty$:

$$\{z \in \mathbb{R}^N, \|z\|_p \leq 1\} = \{z \in \mathbb{R}^N : \exists t \in \mathcal{T} : z^T S_k z \equiv z_k^2 \leq t_k, k \leq K := N\},$$
$$\mathcal{T} = \{t \in \mathbb{R}^N_+ : \|t\|_{p/2} \leq 1\}$$

♠ **Ellitope** $\mathcal{X}$ is a set represented as linear image of a basic ellitope $\mathcal{Z}$:

$$\mathcal{X} = \{x : \exists z \in \mathcal{Z} : x = Pz\}$$
$$\mathcal{Z} = \{z \in \mathbb{R}^N : \exists t \in \mathcal{T} : z^T S_k z \leq t_k, 1 \leq k \leq K\}$$

♠ **Basic spectratope** in $\mathbb{R}^N$ is a *bounded* set $\mathcal{Z}$ given by representation

$$\mathcal{Z} = \{z \in \mathbb{R}^N : \exists t \in \mathcal{T} : S_k^2[z] \preceq t_k I_{d_k}, 1 \leq k \leq K\}$$

where

- $S_k[z] = \sum_{j=1}^N z_j S^{kj}$ is a $d_k \times d_k$ *symmetric* matrix linearly depending on z
- $\mathcal{T} \subset \mathbb{R}^K_+$ is as in the definition of ellitope.

♠ **Example:** Matrix box $\{z \in \mathbb{R}^{p\times q} : \|z\|_{2,2} \leq 1\}$ ($\|\cdot\|_{2,2}$ – spectral norm):

$$\{z \in \mathbb{R}^{p\times q} : \|z\|_{2,2} \leq 1\}$$
$$= \{z \in \mathbb{R}^{p\times q} : \exists t \in [0,1] : \left[\begin{array}{c|c} & z \\ \hline z^T & \end{array}\right]^2 \preceq t I_{p+q}\}.$$

♠ **Spectratope** $\mathcal{X}$ is a set represented as linear image of a basic spectratope $\mathcal{Z}$:

$$\mathcal{X} = \{x : \exists z \in \mathcal{Z} : x = Pz\}$$
$$\mathcal{Z} = \{z \in \mathbb{R}^N : \exists t \in \mathcal{T} : S_k^2[z] \preceq t_k I_{d_k}, 1 \leq k \leq K\}$$

♠ **Fact:** *Every ellitope is a spectratope.* Indeed, if $S_k \succeq 0$, then $S_k = \sum_{j=1}^{r_k} f_{kj} f_{kj}^T \Rightarrow$

$$\begin{array}{l} \{z : \exists t \in \mathcal{T} : z^T S_k z \leq t_k, k \leq K\} \\ = \{z : \exists t \in \mathcal{T}^+ : S_{kj}^2[z] := [f_{kj}^T z]^2 \preceq t_{kj} I_1, j \leq r_k, k \leq K\}, \\ \mathcal{T}^+ = \{\{t_{kj} \geq 0\} : \exists t \in \mathcal{T} : \sum_{j=1}^{r_k} t_{kj} \leq t_k, k \leq K\} \end{array}$$

♠ **Fact:** *Ellitopes/Spectratopes admit fully algorithmic calculus:* nearly all operations preserving "built-in" properties of these sets – convexity, compactness and symmetry w.r.t. the origin, like taking

- finite intersections,
- direct products,
- arithmetic sums,
- linear images,
- inverse images under linear embeddings,

as applied to ellitopes/spectratopes, result in the sets of the same type, with ellitopic/spectratopic representation of the result readily given by respective representations of the operands.

♠ **Note:** *In the main body of the talk, we focus on ellitopes, outlining the extensions to spectratopes at the end.*

Semidefinite Relaxation on Ellitopes

♣ **Standard Semidefinite Relaxation** is aimed at computationally efficient upper-bounding the maximum of quadratic form over a set $\mathcal{Y}$ given by a bunch of quadratic constraints.

♠ In the case of problem of the form

$$\mathrm{Opt}_* = \max_y \left\{ y^T B y : y^T A_k y \leq a_k, k \leq K \right\}$$

SDP relaxation works as follows:

• We observe that whenever $\lambda \in \mathbb{R}^K_+$, we have for feasible y

$$y^T [\sum_k \lambda_k A_k] y \leq \sum_k \lambda_k a_k$$

$\Rightarrow$ Whenever $\lambda \geq 0$ is such that $B \preceq \sum_k \lambda_k A_k$, we have

$$y^T B y \leq \sum_k \lambda_k a_k$$

for all feasible $y \Rightarrow$

$$[\mathrm{Opt}_* \leq] \ \mathrm{Opt} = \min_\lambda \left\{ \sum_k a_k \lambda_k : \lambda \geq 0, B \preceq \sum_k \lambda_k A_k \right\}.$$

$$\boxed{\mathrm{Opt}_* = \max_{y\in\mathcal{Y}} y^T B y}$$

♠ *When $\mathcal{Y}$ is an ellitope:*

$$\mathcal{Y} = \{y : \exists t \in \mathcal{T}, z : y = Pz, z^T S_k z \le t_k, k \le K\}$$

SDP relaxation can be implemented as follows:

• *Let $\lambda \in \mathbb{R}^K_+$ be such that $\widehat{B} := P^T B P \preceq \sum_k \lambda_k S_k$. Whenever $y \in \mathcal{Y}$, $y = Pz$ with $z^T S_k z \le t_k$, $k \le K$, for some $t \in \mathcal{T}$, whence*

$$y^T B y = z^T \widehat{B} z \le z^T \left[\sum_k \lambda_k S_k\right] z \le \sum_k \lambda_k t_k \le \phi_{\mathcal{T}}(\lambda),$$
$$\phi_{\mathcal{T}}(\lambda) := \max_{t\in\mathcal{T}} t^T \lambda$$

$\Rightarrow \mathrm{Opt}_* :=\le \mathrm{Opt} = \min_\lambda \left\{\phi_{\mathcal{T}}(\lambda) : \lambda \ge 0, \widehat{B} \preceq \sum_k \lambda_k S_k\right\}$.

$$\boxed{\begin{array}{rcl}\mathrm{Opt}_* &=& \max_y\left\{y^TBy : \exists t\in\mathcal{T}, z : y=Pz, z^TS_kz\le t_k, k\le K\right\}\\ \mathrm{Opt} &=& \min_\lambda\left\{\phi_{\mathcal{T}}(\lambda) : \lambda\ge 0, \widehat{B}\preceq\sum_k\lambda_kS_k\right\}\end{array}}$$

♠ **Theorem** [Ju&N,'16] *In the ellitopic case, SDP relaxation is reasonably tight:*

$$\mathrm{Opt}_*\le\mathrm{Opt}\le 3\ln(\sqrt{3}K)\mathrm{Opt}_*$$

Proof. Left inequality was already verified. Let

$$\mathbf{T}=\{[t;\tau]:\tau>0, t/\tau\in\mathcal{T}\}\cup\{0\}$$

be the conic hull of $\mathcal{T}$. It is easily seen that $\mathbf{T}$ is a regular (closed, convex, pointed and with a nonempty interior) cone with the dual cone

$$\mathbf{T}_*:=\{[g;s]:[g;s]^T[t;\tau]\ge 0\,\forall[t;\tau]\in\mathbf{T}\}=\{[g;s]:s\ge\phi_{\mathcal{T}}(-g)\}$$

$\Rightarrow$ Opt is the optimal value in the (strictly feasible and solvable) conic problem:

$$\mathrm{Opt}=\min_{\lambda,s}\left\{s:\lambda\ge 0,\widehat{B}\preceq\sum_k\lambda_kS_k,[-\lambda;s]\in\mathbf{T}_*\right\}\quad(*)$$

$\Rightarrow$ Opt is the optimal value in the solvable dual to $(*)$ problem:

$$\begin{array}{rcl}\mathrm{Opt} &=& \max\limits_{Z,[t;\tau],\mu}\left\{\mathrm{Tr}(\widehat{B}Z): \begin{array}{r}Z\succeq 0,\mu\ge 0,[t;\tau]\in\mathbf{T}\\ \sum_k[\mathrm{Tr}(S_kZ)-t_k+\mu_k]\lambda_k+\tau s=s\\ \forall(\lambda,s)\end{array}\right\}\\ &=& \max\limits_{Z,t}\left\{\mathrm{Tr}(\widehat{B}Z):t\in\mathcal{T},Z\succeq 0,\mathrm{Tr}(S_kZ)\le t_k,k\le K\right\}\\ &=& \mathrm{Tr}(\widehat{B}Z_*)\quad[Z_*\succeq 0,\exists t^*\in\mathcal{T}:\mathrm{Tr}(S_kZ_*)\le t^*_k,k\le K]\end{array}$$

$$\boxed{\text{Opt} = \text{Tr}(\widehat{B}Z_*) \quad [Z_* \succeq 0, \exists t^* \in \mathcal{T} : \text{Tr}(S_k Z_*) \le t^*_k, k \le K]}$$

• Let

$$\widetilde{B} := Z_*^{1/2}\widehat{B}Z_*^{1/2} = U\text{Diag}\{\mu\}U^T \qquad [U \text{ is orthogonal}]$$

and let $\widetilde{S}_k = U^T Z_*^{1/2} S_k Z_*^{1/2} U$, so that

$$0 \preceq \widetilde{S}_k, \ \text{Tr}(\widetilde{S}_k) = \text{Tr}(Z_*^{1/2} S_k Z_*^{1/2}) = \text{Tr}(S_k Z_*) \le t^*_k.$$

Let ζ be Rademacher random vector (independent entries taking values ± 1 with probability $1/2$), and let $\xi = Z_*^{1/2}U\zeta$. We have

$$\begin{array}{rcl} \mathbf{E}\{\xi\xi^T\} & = & \mathbf{E}\{Z_*^{1/2}U\zeta\zeta^T U^T Z_*^{1/2}\} = Z_* \\ \xi^T\widehat{B}\xi & = & \zeta^T U^T Z_*^{1/2}\widehat{B}Z_*^{1/2}U\zeta = \zeta^T U^T \widetilde{B} U\zeta \\ & = & \zeta^T \text{Diag}\{\mu\}\zeta = \sum_i \mu_i = \text{Tr}(\widetilde{B}) = \text{Tr}\widehat{B}Z_*) = \text{Opt} \\ \xi^T S_k \xi & = & \zeta^T U^T Z_*^{1/2} S_k Z_*^{1/2} U\zeta = \zeta^T \widetilde{S}_k \zeta \end{array}$$

• When k is such that $t^*_k = 0$, we have $\widetilde{S}_k = 0 \Rightarrow \xi^T S_k \xi \equiv 0$

• When k is such that $t^*_k > 0$, we have $\text{Tr}(\widetilde{S}_k/t^*_k) \le 1 \Rightarrow$

$$\left[\mathbf{E}\left\{\exp\left\{\frac{\xi^T S_k \xi}{3t^*_k}\right\}\right\} = \right] \mathbf{E}\left\{\exp\left\{\frac{\zeta^T \widetilde{S}_k \zeta}{3t^*_k}\right\}\right\} \le \sqrt{3}$$

due to

Mini-Lemma: *Let Q be positive semidefinite $N \times N$ matrix with trace ≤ 1 and ζ be N-dimensional Rademacher random vector. Then*

$$\mathbf{E}\left\{\exp\left\{\zeta^T Q\zeta/3\right\}\right\} \le \sqrt{3}.$$

$$\begin{array}{rcl} \text{Opt} & := & \max\limits_{Z,t} \left\{ \text{Tr}(\widehat{B}Z) : t \in \mathcal{T}, Z \succeq 0, \text{Tr}(S_k Z) \leq t_k, k \leq K \right\} \\ & \geq & \text{Opt}_* := \max_z \left\{ z^T \widehat{B} z : \exists t \in \mathcal{T} : z^T S_k z \leq t_k, k \leq K \right\} \end{array}$$

$$\xi^T \widehat{B} \xi \equiv \text{Opt} \ \& \ \underbrace{\xi^T S_k \xi \equiv 0 \text{ if } t_k^* = 0 \ \& \ \mathbf{E}\{\exp\{\frac{\xi^T S_k \xi}{3t_k^*}\}\} \leq \sqrt{3} \text{ if } t_k^* > 0}_{(*)}$$

$\Rightarrow$ [by $(*)$] $\quad \text{Prob}\{\exists k : \xi^T S_k \xi > 3\ln(\sqrt{3}K) t_k^*\} < 1$

$\Rightarrow \exists \bar{\xi} : \bar{\xi}^T S_k \bar{\xi} \leq 3\ln(\sqrt{3}K) t_k^*, k \leq K \ \& \ \bar{\xi}^T \widehat{B} \bar{\xi} = \text{Opt}$

$\Rightarrow$ setting $z = \bar{\xi}/\sqrt{3\ln(\sqrt{3}K)}$, we get

$$z^T S_k z \leq t_k^*, \ k \leq K \ \& \ z^T \widehat{B} z = \text{Opt}/[3\ln(\sqrt{3}K)]$$

$\Rightarrow \text{Opt} \leq 3\ln(\sqrt{3}K)\text{Opt}_*$ $\square$

Proof of Mini-Lemma: Let $Q=\sum_i \sigma_i f_i f_i^T$ be the eigenvalue decomposition of Q, so that $f_i^T f_i = 1$, $\sigma_i \geq 0$, and $\sum_i \sigma_i \leq 1$. The function

$$f(\sigma_1,...,\sigma_N) = \mathbf{E}\left\{\mathrm{e}^{\frac{1}{3}\sum_i \sigma_i \zeta^T f_i f_i^T \zeta}\right\}$$

is convex on the simplex $\{\sigma \geq 0, \sum_i \sigma_i \leq 1\}$ and thus attains it maximum over the simplex at a vertex, implying that for some $f = f_i$, $f^T f = 1$, it holds

$$\mathbf{E}\{\mathrm{e}^{\frac{1}{3}\zeta^T Q \zeta}\} \leq \mathbf{E}\{\mathrm{e}^{\frac{1}{3}(f^T\zeta)^2}\}.$$

Let $\xi \sim \mathcal{N}(0,1)$ be independent of ζ. We have

$$\begin{array}{l}
\mathbf{E}_\zeta\left\{\exp\{\frac{1}{3}(f_i^T\zeta)^2\}\right\} = \mathbf{E}_\zeta\left\{\mathbf{E}_\xi\left\{\exp\{[\sqrt{2/3}f^T\zeta]\xi\}\right\}\right\}\\
= \mathbf{E}_\xi\left\{\mathbf{E}_\zeta\left\{\exp\{[\sqrt{2/3}f^T\zeta]\xi\}\right\}\right\} = \mathbf{E}_\xi\left\{\prod_{j=1}^N \mathbf{E}_\zeta\left\{\exp\{\sqrt{2/3}\xi f_j \zeta_j\}\right\}\right\}\\
= \mathbf{E}_\xi\left\{\prod_{j=1}^N \cosh(\sqrt{2/3}\xi f_j)\right\} \leq \mathbf{E}_\xi\left\{\prod_{j=1}^N \exp\{\xi^2 f_j^2/3\}\right\}\\
= \mathbf{E}_\xi\left\{\exp\{\xi^2/3\}\right\} = \sqrt{3}
\end{array}$$

□

What actually happened?

$$\boxed{\mathrm{Opt}_* = \max_{z,t}\left\{z^T\widehat{B}z : t\in\mathcal{T}, z^TS_kz\leq t_k, k\leq K\right\}} \qquad (*)$$

♣ The dual form

$$\mathrm{Opt} = \max_{Z,t}\left\{\mathrm{Tr}(\widehat{B}Z) : Z\succeq 0, t\in\mathcal{T}, \mathrm{Tr}(S_kZ)\leq t_k, k\leq K\right\} \qquad (D)$$

of SDP relaxation

$$\mathrm{Opt} = \min_{\lambda}\left\{\phi_{\mathcal{T}}(\lambda) : \lambda\geq 0, \widehat{B}\preceq \sum\nolimits_k \lambda_k S_k\right\} \qquad (P)$$

of $(*)$ can be interpreted as follows:

♣ We pass from *deterministic* feasible solutions (z,t) to $(*)$ to *random solutions* $(\widetilde{z},\widetilde{t})$ satisfying the constraints *at average:*

$$\mathbf{E}\{\widetilde{t}\}\in\mathcal{T},\ \mathbf{E}\{\widetilde{z}^TS_k\widetilde{z}\}\leq\mathbf{E}\{\widetilde{t}_k\}, k\leq K$$

and maximize over these random solutions the *expected value* $\mathbf{E}\{\widetilde{z}^T\widehat{B}\widetilde{z}\}$ of the objective.

Note: *What matters in the latter problem, is the expectation t of $\widetilde{t}$ and the covariance matrix Z of $\widetilde{z}$, and in terms of t, Z, the problem is exactly (D).*

• The advantage of "average" interpretation of (D) is that *given an optimal solution to (D), we can build* (in many ways!) *associated random solution $\widetilde{z},\widetilde{t}$ and then "correct" realizations of $\widetilde{z},\widetilde{t}$ to make the corrections feasible for $(*)$. With luck, we can control the price of the correction in terms of the actual objective, thus quantifying the "gap" between Opt and Opt_*.*

$$\begin{array}{rcl} \mathrm{Opt}_* & = & \max_{z,t}\left\{z^T\widehat{B}z : t\in\mathcal{T}, z^TS_kz\le t_k, k\le K\right\}\\ \mathrm{Opt} & = & \max_{Z,t}\left\{\mathrm{Tr}(\widehat{B}Z) : Z\succeq 0, t\in\mathcal{T}, \mathrm{Tr}(S_kZ)\le t_k,\ k\le K\right\}\\ & \ge & \mathrm{Opt}_* \end{array}$$

♠ In our analysis of the gap between Opt_* and Opt,

• the random solution was ξ, t^*, the objective at this solution was *identically equal to* Opt, and we ensured that

$$\mathbf{E}\{\xi^T S_k \xi\} \le t_k^*,\ k \le K$$

• correction was of the form

$$\xi \mapsto z = \left[\min_{k:t_k^*>0} \frac{t_k^*}{\xi^T S_k \xi}\right]^{1/2} \xi \Rightarrow z^T\widehat{B}z = \left[\min_{k:t_k^*>0} \frac{t_k^*}{\xi^T S_k \xi}\right] \mathrm{Opt}$$

• we show that the random "price of correction" $\min\limits_{k:t_k^*>0} \frac{t_k^*}{\xi^T S_k \xi}$ with positive probability is $\ge \frac{1}{3\ln(\sqrt{3}K)}$

$$\Rightarrow \mathrm{Opt} \le 3\ln(\sqrt{3}K)\mathrm{Opt}_*$$

♠ **Fact:** All known to us approximation results for SDP relaxations utilize the above strategy *"find good on average random solution and correct its realizations."*

Executive Summary on Conic Programming

♠ **Conic program** is optimization program of the form

$$\mathrm{Opt}(P) = \min_x \left\{c^T x : A_i x - b_i \in \mathbf{K}_i, i \le m, Px = p\right\} \qquad (P)$$

where $\mathbf{K}_i$ are *regular* (convex, closed, pointed, and with a nonempty interior) cones in $\mathbb{R}^{n_i}$.

♠ **Dual to** (P) **program** stems from the desire to lower-bound $\mathrm{Opt}(P)$ and is as follows:

• We equip the conic constraints $A_i x - b_i \in \mathbf{K}_i$ with *Lagrange multipliers* λ_i belonging to the cones

$$\mathbf{K}_i^* = \{\lambda : \lambda^T y \ge 0\ \forall y \in \mathbf{K}_i\}$$

dual to $\mathbf{K}_i$, and equip the equality constraints $Px = p \in \mathbb{R}^k$ with Lagrange multiplier $\mu \in \mathbb{R}^k$.

• Summing up the constraints in (P) with weights λ_i, μ, we get *aggregated constraint*

$$\left[\sum_i A_i^T \lambda_i + P^T \mu\right]^T x \ge \sum_i b_i^T \lambda_i + p^T \mu \qquad (*)$$

which is a consequence of the constraints in (P)

$\Rightarrow$ *Whenever the left hand side in the aggregated constraint identically in x is $c^T x$, the right hand side in $(*)$ is a lower bound on $\mathrm{Opt}(P)$.* The dual problem

$$\mathrm{Opt}(D) = \max_{\lambda_i, \mu} \left\{ \sum_i b_i^T \lambda_i + p^T \mu : \begin{array}{l} \lambda_i \in \mathbf{K}_i^*, i \le m \\ \sum_i A_i^T \lambda_i + P^T \mu = c \end{array} \right\}$$

is to find the best possible bound of this type.

$$\begin{array}{rcl} \mathrm{Opt}(P) & = & \min\limits_{x}\left\{c^Tx : A_ix - b_i \in \mathbf{K}_i, Px = p\right\} \qquad (P) \\ \mathrm{Opt}(D) & = & \max\limits_{\lambda,\mu}\left\{\sum_i b_i^T\lambda_i + p^T\mu : \begin{array}{l}\lambda_i \in \mathbf{K}_i^*, i \leq m \\ \sum_i A_i^T\lambda_i + P^T\mu = e\end{array}\right\} \qquad (D) \end{array}$$

♠ A conic problem is called *strictly feasible*, if it admits a feasible solution for which the left hand sides of all conic constraints belong to the *interiors* of the right hand side cones.

♠ **Conic Duality Theorem:**

[symmetry] *Conic duality is symmetric: the dual problem (D) is a conic one, and its dual is (equivalent to) the primal problem (P).*

[weak duality] *One always have* $\mathrm{Opt}(D) \leq \mathrm{Opt}(P)$

[strong duality] *Let one of the problems (P), (D) be strictly feasible and bounded. Then the other problem is solvable, and optimal values are equal to each other:* $\mathrm{Opt}(D) = \mathrm{Opt}(P)$.

Near-optimality of linear estimates: Bounded noise

♣ **Situation:** Given observation $\omega = Ax + \eta$ of *unknown* signal x *known to belong to a given signal set* $\mathcal{X}$, we want to recover Bx. All we know about the noise is $\eta \in \mathcal{H}$, with a known and bounded set $\mathcal{H}$.

We define the risk of an estimate $\omega \mapsto \widehat{x}(\omega)$ as

$$\mathrm{Risk}_{\|\cdot\|,\mathcal{H}}[\widehat{x}|\mathcal{X}] = \sup_{x\in\mathcal{X},\eta\in\mathcal{H}} \|Bx - \widehat{x}(Ax+\eta)\|$$

♠ **Assumptions:** $\mathcal{X}, \mathcal{H}$ *are ellitopes, and the unit ball*

$$\mathcal{B}_* = \{u : \|u\|_* \leq 1\}$$

of the norm conjugate to $\|\cdot\|$ *is a basic ellitope*, as is the case when

$$\|\cdot\| = \|\cdot\|_p,\ 1 \leq p \leq 2.$$

♠ **Immediate observation:** *The situation in question reduces to the one with no noise.*

Indeed, we can think that the signal underlying observation is $[x;\eta]$ rather than x. In terms of this signal,

- the observation is $\bar{A}[x;\eta] = Ax + \eta$,
- the quantity to be recovered is $\bar{B}[x;\eta] = Bx$,
- the signal $[x;\eta]$ is known to belong to $\mathcal{Y} := \mathcal{X} \times \mathcal{H}$, which is an ellitope,
- the performance of a candidate estimate is quantified by the worst-case risk

$$\mathrm{Risk}_{\|\cdot\|}[\widehat{x}|\mathcal{Y}] = \sup_{y=[x;\eta]\in\mathcal{Y}} \|\bar{B}y - \widehat{x}(\bar{A}y)\| \qquad [= \mathrm{Risk}_{\|\cdot\|,\mathcal{H}}[\widehat{x}|\mathcal{X}]]$$

⇒ *We assume from now on that there is no observation noise:*

$$\omega = Ax,\ x \in \mathcal{X},$$

$\mathcal{X}$ is an ellitope, and the risk is defined as

$$\mathrm{Risk}_{\|\cdot\|}[\widehat{x}|\mathcal{X}] = \sup_{x\in\mathcal{X}} \|Bx - \widehat{x}(Ax)\|.$$

We further lose nothing when assuming that $\mathcal{X}$ is a basic ellitope:

$$\mathcal{X} = \{x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \le t_k, k \le K\}.$$

♣ **Building linear estimate.** To get the minimum risk *linear* estimate $\widehat{x}_H(\omega) = H^T\omega$, we need to solve the optimization problem

$$\mathrm{Opt}_* = \min_H \left\{\Phi_*(H) := \max_{x\in\mathcal{X}} \|Bx - H^T Ax\|\right\} \qquad (!)$$

Difficulty: While $\Phi_*(H)$ is convex (as the supremum of a family of convex functions of H), this function could be difficult to compute

⇒ *in general,* (!) *is intractable.*

Nearly the only known cases where $\mathcal{X}$ is an ellitope and (!) is tractable are those of

- ellipsoid $\mathcal{X}$ and Euclidean norm $\|\cdot\|$
- $\|\cdot\| = \|\cdot\|_\infty$.

$$\begin{array}{rcl} \multicolumn{3}{c}{\mathrm{Opt}_* = \min_H \left\{\Phi_*(H) := \max_{x\in\mathcal{X}} \|Bx - H^TAx\|\right\}} \\ \mathcal{X} &=& \{x\in\mathbb{R}^n : \exists t\in\mathcal{T} : x^TS_kx \leq t_k, k\leq K\} \\ \mathcal{B}_* &:=& \{u : \|u\|_* \leq 1\} = \{u\in\mathbb{R}^\nu : \exists r\in\mathcal{R} : u^TR_\ell u \leq r_\ell, \ell\leq L\} \end{array}$$

♠ **Observation:** $\Phi_*(H)$ is the maximum of a quadratic form over an ellitope:

$\|v\| = \max_{u\in\mathcal{B}_*} u^Tv \Rightarrow$

$$\begin{array}{rcl} \Phi_*(H) &=& \max_{[u;x]\in\mathcal{B}_*\times\mathcal{X}} u^T[B - H^TA]x \\ &=& \max_{[u;x]\in\mathcal{B}_*\times\mathcal{X}} [u;x]^TW(H)[u;x], \\ W(H) &=& \left[\begin{array}{c|c} & \frac{1}{2}[B - H^TA] \\ \hline \frac{1}{2}[B^T - A^TH] & \end{array}\right] \end{array}$$

$\Rightarrow$ by SDP relaxation, $\Phi_*(H)$ *admits an efficiently computable convex upper bound*

$$\Phi(H) = \min_{\lambda,\mu}\left\{\phi_{\mathcal{T}}(\lambda) + \phi_{\mathcal{R}}(\mu) : \begin{array}{c} \lambda\geq 0, \mu\geq 0 \\ \left[\begin{array}{c|c} \sum_\ell \mu_\ell R_\ell & \frac{1}{2}[B - H^TA] \\ \hline \frac{1}{2}[B^T - A^TH] & \sum_k \lambda_k S_k \end{array}\right] \end{array}\right\}$$
$$\left[\phi_{\mathcal{T}}(\lambda) = \max_{t\in\mathcal{T}} t^T\lambda, \phi_{\mathcal{R}}(\mu) = \max_{r\in\mathcal{R}} r^T\mu\right]$$

$\Rightarrow$ *We can approximate intractable problem of building the best linear estimate with efficiently solvable problem*

$$\mathrm{Opt} = \min_{\lambda,\mu,H}\left\{\phi_{\mathcal{T}}(\lambda) + \phi_{\mathcal{R}}(\mu) : \begin{array}{c} \lambda\geq 0, \mu\geq 0 \\ \left[\begin{array}{c|c} \sum_\ell \mu_\ell R_\ell & \frac{1}{2}[B - H^TA] \\ \hline \frac{1}{2}[B^T - A^TH] & \sum_k \lambda_k S_k \end{array}\right] \end{array}\right\}$$

The H-component H_* of optimal solution to this problem yields linear estimate $\widehat{x}_{H_*}(\omega) = H_*^T\omega$ satisfying

$$\mathrm{Risk}_{\|\cdot\|}[\widehat{x}_{H_*}|\mathcal{X}] \leq \mathrm{Opt} \qquad [\leq 3\ln(\sqrt{3}[K+L])\mathrm{Opt}_*]$$

♣ **Theorem** [Ju&N,'17] *The linear estimate $\widehat{x}_{H_*}$ yielded by (efficiently computable) optimal solution H_* to the above problem is near-optimal:*

$$\mathrm{Risk}_{\|\cdot\|}[\widehat{x}_{H_*}|\mathcal{X}] \leq \mathrm{Opt} \leq 3\ln(\sqrt{3}[K+L])\mathrm{Risk}^*_{\|\cdot\|}[\mathcal{X}],$$

where

$$\mathrm{Risk}^*_{\|\cdot\|}[\mathcal{X}] = \inf_{\widehat{x}(\cdot)} \mathrm{Risk}_{\|\cdot\|}[\widehat{x}|\mathcal{X}],$$

inf *being taken over all estimates, linear and nonlinear alike, is the minimax optimal risk.*

♠ **Sketch of the proof:**

A. Consider the quantity

$$\mathfrak{R} = \max_x \{\|Bx\| : Ax = 0, x \in \mathcal{X}\}.$$

Claim: *$\mathfrak{R}$ is a lower bound on minimax optimal risk $\mathrm{Risk}^*_{\|\cdot\|}[\mathcal{X}]$.*

Indeed,

- $\exists \bar{x} \in \mathcal{X} : A\bar{x} = 0$ & $\|B\bar{x}\| = \mathfrak{R}$

$\Rightarrow$ observation $\omega = 0$ may come from signals $\bar{x}_\pm := \pm\bar{x} \in \mathcal{X}$

$\Rightarrow$ minimax risk cannot be less that $\mathfrak{R} = \frac{1}{2}\|B\bar{x}_+ - B\bar{x}_-\|$.

B. Let E be a matrix with trivial kernel and columns spanning $\mathrm{Ker}A$. We have

$$\mathfrak{R} = \max_y \{\|BEy\| : y \in \mathcal{Y}\}, \ \mathcal{Y} = \{y : Ey \in \mathcal{X}\},$$

$\Rightarrow \mathfrak{R} = \max_{u \in \mathcal{B}_*, y \in \mathcal{Y}} u^T[BE]y$ is the maximum of a quadratic form over the ellitope $\mathcal{B}_* \times \mathcal{Y}$
$\Rightarrow \mathfrak{R}$ can be tightly upper-bounded by semidefinite relaxation. On a closest inspection (heavily utilizing conic duality), *this bound turns out to be* $\geq \mathrm{Opt}$, where Opt is the SDP relaxation bound on the risk of $\widehat{x}_{H_*}$
$\Rightarrow \mathrm{Opt}$ tightly upper-bounds $\mathfrak{R}$ and thus – the minimal optimal risk.

♠ **Note:** Theorem is nice but not too important, since we can easily build a nearly optimal efficiently computable *nonlinear* estimate, namely, as follows:

> *Given observation $\omega = Ax$ with unknown $x \in \mathcal{X}$, we solve convex feasibility problem*
>
> $$\textit{find } \bar{x} \in \mathcal{X} : A\bar{x} = \omega$$
>
> *and estimate Bx by $B\bar{x}$, where $\bar{x}$ is (any) solution to the feasibility problem.*

This estimate is efficiently computable under much weaker assumptions than those underlying Theorem, and *always is minimax optimal within factor 2.*

Near-optimality of linear estimates: Random noise

♣ **Situation:** Given observation $\omega = Ax + \eta$ of *unknown* signal x *known to belong to a given signal set* $\mathcal{X}$, we want to recover Bx. All we know about the noise is that η is random with covariance matrix

$$\mathrm{Cov}[\eta] = \mathbf{E}\{\eta\eta^T\}$$

belonging to a given convex compact subset Θ of the *interior* of positive semidefinite cone.

We define the risk of an estimate $\omega \mapsto \widehat{x}(\omega)$ as

$$\mathrm{Risk}_{\|\cdot\|,\Theta}[\widehat{x}|\mathcal{X}] = \sup_{\substack{x\in\mathcal{X}\\ \eta:\mathrm{Cov}[\eta]\in\Theta}} \mathbf{E}\|\widehat{x}(Ax+\eta) - Bx\|$$

♠ **Assumptions:** *$\mathcal{X}$ and the unit ball $\mathcal{B}_*$ of the norm $\|\cdot\|_*$ conjugate to $\|\cdot\|$ are ellitopes.*

For example, we can handle the case $\|\cdot\| = \|\cdot\|_p$, $1 \le p \le 2$.

• On a simple inspection, we lose nothing when assuming that $\mathcal{X}$ is a basic ellitope:

$$\mathcal{X} = \{x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \le t_k,\, k \le K\}$$

while

$$\begin{array}{c}\mathcal{B}_* := \{u : \|u\|_* \le 1\} = \{u \in \mathbb{R}^\nu : \exists y \in \mathcal{Y} : u = My\},\\ \mathcal{Y} = \{y \in \mathbb{R}^N : \exists r \in \mathcal{R} : y^T R_\ell y \le r_\ell,\, \ell \le L\}.\end{array}$$

Building “good” linear estimate

$$\begin{array}{rcl} \mathcal{X} & = & \{x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k,\, k \leq K\} \\ \mathcal{B}_* & := & \{u : \|u\|_* \leq 1\} = \{u \in \mathbb{R}^\nu : \exists y \in \mathcal{Y} : u = My\} \\ \mathcal{Y} & = & \{y \in \mathbb{R}^N : \exists r \in \mathcal{R} : y^T R_\ell y \leq r_\ell,\, \ell \leq L\} \end{array}$$

♠ **Risk Analysis:** Let $\widehat{x}_H(\omega) = H^T\omega$ be a candidate linear estimate. Let us upper-bound its risk:

$$\begin{array}{l} \mathrm{Risk}_{\|\cdot\|,\Theta}[\widehat{x}_H|\mathcal{X}] \\ \quad = \sup\limits_{\substack{x\in\mathcal{X} \\ \eta:\mathrm{Cov}[\eta]\in\Theta}} \mathbf{E}\left\{\|Bx - H^T(Ax+\eta)\|\right\} \\ \quad \leq \sup\limits_{\substack{x\in\mathcal{X} \\ \eta:\mathrm{Cov}[\eta]\in\Theta}} \mathbf{E}\left\{\|[B-H^TA]x\| + \|H^T\eta\|\right\} \\ \quad = \underbrace{\max\limits_{x\in\mathcal{X}} \|[B-H^TA]x\|}_{\Phi_*(H)} + \underbrace{\sup\limits_{\eta:\mathrm{Cov}[\eta]\in\Theta} \mathbf{E}\left\{\|H^T\eta\|\right\}}_{\Psi_*(H)} \end{array}$$

• Our ideal goal would be to select H as an optimal solution to the optimization problem

$$\min_H \left\{\Phi_*(H) + \Psi_*(H)\right\};$$

however, functions Φ_* and Ψ_*, while convex, can be difficult to compute

⇒ *We indent to replace Φ_*, Ψ_* with their efficiently computable convex upper bounds.*

$$
\boxed{\begin{array}{rcl}
\mathcal{X} &=& \{x\in\mathbb{R}^n : \exists t\in\mathcal{T} : x^TS_kx\le t_k,\, k\le K\}\\
\mathcal{B}_* &:=& \{u : \|u\|_*\le 1\} = \{u\in\mathbb{R}^\nu : \exists y\in\mathcal{Y} : u=My\},\\
\mathcal{Y} &=& \{y\in\mathbb{R}^N : \exists r\in\mathcal{R} : y^TR_\ell y\le r_\ell,\, \ell\le L\}
\end{array}}
$$

♠ **Upper-bounding Φ_*.** We already know how to upper-bound Φ_*:

$$
\begin{array}{rcl}
\Phi_*(H) &=& \max_{x\in\mathcal{X}}\|[B-H^TA]x\|\\
&=& \max_{[u;x]\in\mathcal{B}_*\times\mathcal{X}} u^T[B-H^TA]x\\
&=& \max_{[y;x]\in\mathcal{Y}\times\mathcal{X}} y^TM^T[B-H^TA]x
\end{array}
$$

⇒ [SDP relaxation]

$$
\begin{array}{rcl}
\Phi_*(H) &\le& \Phi(H) = \min\limits_{\lambda,\mu}\left\{\phi_{\mathcal{T}}(\lambda)+\phi_{\mathcal{R}}(\mu) : \lambda\ge 0, \mu\ge 0,\right.\\
&& \left.\left[\begin{array}{c|c}\sum_\ell \mu_\ell R_\ell & \frac{1}{2}M^T[B-H^TA]\\ \hline \frac{1}{2}[B^T-AH^T]M & \sum_k\lambda_kS_k\end{array}\right]\succeq 0\right\}\\
&\le& 3\ln(\sqrt{3}[K+L])\Phi(H).
\end{array}
$$

$$\begin{array}{rcl}\mathcal{B}_* & := & \{u : \|u\|_* \leq 1\} = \{u \in \mathbb{R}^\nu : \exists y \in \mathcal{Y} : u = My\}, \\ \mathcal{Y} & = & \{y \in \mathbb{R}^N : \exists r \in \mathcal{R} : y^T R_\ell y \leq r_\ell, \, \ell \leq L\}\end{array}$$

♠ **Upper-bounding Ψ_*.**

Lemma: *Let* $Q = \mathrm{Cov}[\eta]$. *Then*

$$\mathbf{E}\left\{\|H^T\eta\|\right\} \leq \min_{G,\mu}\left\{\phi_{\mathcal{R}}(\mu) + \mathrm{Tr}(QG) : \mu \geq 0, \left[\begin{array}{c|c}\sum_\ell \mu_\ell R_\ell & \frac{1}{2}M^T H^T \\ \hline \frac{1}{2}HM & G\end{array}\right] \succeq 0\right\} \qquad (*)$$

As a result,

$$\begin{array}{rcl}\Psi_*(H) & \leq & \Psi(H) := \min\limits_{G,\mu}\left\{\phi_{\mathcal{R}}(\mu) + \Gamma(G) : \mu \geq 0, \left[\begin{array}{c|c}\sum_\ell \mu_\ell R_\ell & \frac{1}{2}M^T H^T \\ \hline \frac{1}{2}HM & G\end{array}\right] \succeq 0\right\}, \\ \Gamma(G) & = & \max\limits_{Q \in \Theta} \mathrm{Tr}(QG).\end{array}$$

Proof. Let (G, μ) be feasible for $(*)$. By semidefinite constraint, we have $y^T M^T H^T \eta \leq y^T [\sum_\ell \mu_\ell R_\ell] y + \eta^T G \eta \; \forall y, \eta$ $\Rightarrow$

$$\begin{array}{l}\|H^T\eta\| = \max\limits_{u \in \mathcal{B}_*} u^T H^T \eta = \max\limits_{y,r}\left\{[My]^T H^T \eta : r \in \mathcal{R}, y^T R_\ell y \leq r_\ell, \ell \leq L\right\} \\ \quad \leq \max\limits_{y,r}\left\{y^T[\sum_\ell \mu_\ell R_\ell]y + \eta^T G \eta : r \in \mathcal{R}, y^T R_\ell y \leq r_\ell, \ell \leq L\right\} \\ \quad \leq \max\limits_{r \in \mathcal{R}}\left\{\sum_\ell \mu_\ell r_\ell\right\} + \eta^T G \eta = \phi_{\mathcal{R}}(\mu) + \eta^T G \eta.\end{array}$$

$\Rightarrow$ [taking expectation] $\mathbf{E}\{\|H^T\eta\|\} \leq \phi_{\mathcal{R}}(\mu) + \mathrm{Tr}(QG)$. □

♠ **Illustration:** Let $\|\cdot\| = \|\cdot\|_p$ with $1 \leq p \leq 2$ and $\Theta = \{Q\}$. *The yielded by our construction upper bound* $\Psi(H)$ *on* $\mathbf{E}\{\|H^T\eta\|_p\}$, $\mathrm{Cov}[\eta] = Q$, *turns out to be*

$$\left\| \left[\|Q^{1/2}\mathrm{Col}_1[H]\|_2; ...; \|Q^{1/2}\mathrm{Col}_\nu[H]\|_2 \right] \right\|_p$$

$$
\begin{array}{rcl}
\mathcal{X} & = & \{x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \le t_k,\, k \le K\} \\
\mathcal{B}_* & := & \{u : \|u\|_* \le 1\} = \{u \in \mathbb{R}^\nu : \exists y \in \mathcal{Y} : u = My\}, \\
\mathcal{Y} & = & \{y \in \mathbb{R}^N : \exists r \in \mathcal{R} : y^T R_\ell y \le r_\ell,\, \ell \le L\}
\end{array}
$$

♠ **Putting things together:**

Theorem [Ju&N,'17] *Consider convex optimization problem*

$$
\begin{array}{l}
\mathrm{Opt} = \min\limits_{H} \{\Phi(H) + \Psi(H)\} \\
= \min\limits_{H,G,\lambda,\mu,\mu'} \left\{ \phi_{\mathcal{T}}(\lambda) + \phi_{\mathcal{R}}(\mu) + \phi_{\mathcal{R}}(\mu') + \max\limits_{Q \in \Theta} \mathrm{Tr}(QG) : \right. \\
\qquad \lambda \ge 0, \mu \ge 0, \mu' \ge 0 \\
\qquad \left[\begin{array}{c|c} \sum\limits_\ell \mu_\ell R_\ell & \frac{1}{2} M^T [B - H^T A] \\ \hline \frac{1}{2}[B^T - A^T H] M & \sum\limits_k \lambda_k S_k \end{array} \right] \succeq 0 \\
\qquad \left. \left[\begin{array}{c|c} \sum\limits_\ell \mu'_\ell R_\ell & \frac{1}{2} M^T H^T \\ \hline \frac{1}{2} HM & G \end{array} \right] \succeq 0 \right\}
\end{array}
$$

The problem is efficiently solvable, and the linear estimate $\widehat{x}_{H_*}(\omega) = H_*^T \omega$ induced by the H-component of an optimal solution satisfies the risk bound

$$
\mathrm{Risk}_{\|\cdot\|,\Theta}[\widehat{x}_{H_*} | \mathcal{X}] \le \mathrm{Opt}.
$$

Near-Optimality in Gaussian case

$$\begin{array}{rcl}\mathcal{X} & = & \{x\in\mathbb{R}^n : \exists t\in\mathcal{T} : x^TS_kx\leq t_k,\, k\leq K\}\\ \mathcal{B}_* & := & \{u:\|u\|_*\leq 1\}=\{u\in\mathbb{R}^\nu : \exists y\in\mathcal{Y} : u=My\},\\ \mathcal{Y} & = & \{y\in\mathbb{R}^N : \exists r\in\mathcal{R} : y^TR_\ell y\leq r_\ell,\,\ell\leq L\}\end{array}$$

♣ **Theorem** [Ju&N,'17] *The linear estimate $\widehat{x}_{H_*}(\cdot)$ yielded by previous Theorem is "near minimax optimal:" for properly selected matrix $Q\in\Theta$ one has*

$$\begin{array}{l}\mathrm{Risk}_{\|\cdot\|,\Theta}[\widehat{x}_{H_*}|\mathcal{X}]\leq \mathrm{Opt}\\ \qquad\leq O(1)\sqrt{\ln(2L)\ln\left(\frac{2KM_*^2}{\mathrm{RiskOpt}^2_{\|\cdot\|,Q}[\mathcal{X}]}\right)}\mathrm{RiskOpt}_{\|\cdot\|,Q}[\mathcal{X}],\end{array}\qquad (!)$$

where $O(1)$ is an appropriate absolute constant,

$$M_*^2=\max_W\left\{\mathbf{E}_{\zeta\sim\mathcal{N}(0,W)}\{\zeta^TB^T\zeta\} : W\succeq 0, \exists t\in\mathcal{T} : \mathrm{Tr}(WS_k)\leq t_k, k\leq K\right\}$$

and $\mathrm{RiskOpt}_{\|\cdot\|,Q}[\mathcal{X}]$ is the minimax optimal risk of recovering Bx, $x\in\mathcal{X}$, from noisy observation $\omega=Ax+\eta$ with zero mean Gaussian noise $\eta\sim\mathcal{N}(0,Q)$:

$$\mathrm{RiskOpt}_{\|\cdot\|,Q}[\mathcal{X}]=\inf_{\widehat{x}(\cdot)}\sup_{x\in\mathcal{X}}\mathbf{E}_{\eta\sim\mathcal{N}(0,Q)}\left\{\|Bx-\widehat{x}(Ax+\eta)\|\right\},$$

inf *being taken over all estimates $\widehat{x}(\cdot)$, linear and nonlinear alike.*

♠ **Surprise:** *Nonoptimality factor in (!) is "nearly constant" and is independent of interplay between the geometries of $\mathcal{X}$, $\|\cdot\|$, A and B – the entities primarily and heavily responsible for the minimax optimal risk.*

♠ **Sketch of the proof:**

A. By simple saddle point argument, the upper bound Opt on the risk of the optimal linear estimate is *as if* the set Θ of allowed covariance matrices of observation noise was replaced with a properly selected *singleton* $\{Q\} \in \Theta$.

From now on we assume that the observation noise is $\eta \sim \mathcal{N}(0, Q)$.

B. The idea of the proof (originating from M.S. Pinsker (1982) who considered simple case where $\mathcal{X}$ is ellipsoid, $\|\cdot\| = \|\cdot\|_2$, $A = B = I$) is to consider, instead of minimax optimal risk, the optimal *Bayesian* risk

$$\mathrm{RiskB}[W] = \inf_{\widehat{x}(\cdot)} \mathbf{E}_{\eta \sim \mathcal{N}(0,Q), \xi \sim \mathcal{N}(0,W)} \left\{ \|B\xi - \widehat{x}(A\xi + \eta)\| \right\},$$

where Gaussian random signal $\xi \sim \mathcal{N}(0, W)$ is independent of observation noise $\eta \sim \mathcal{N}(0, Q)$, and we are interested in the minimal, over all estimates, *expected* risk, the expectation being taken over both signal and noise.

• Similarly to the Gauss-Markov Theorem, it is easy to prove that the optimal Bayesian risk is achieved, within a moderate absolute constant factor, on a *linear* estimate (conditional expectation of $B\xi$ given $\omega = A\xi + \eta$). As a result,

$$\begin{array}{l} \forall W \succeq 0 \, \exists H_W : \\ \underbrace{\mathbf{E}_{\xi \sim \mathcal{N}(0,W)} \left\{ \|[B - H_W^T A]\xi\| \right\}}_{\text{bias}} + \underbrace{\mathbf{E}_{\eta \sim \mathcal{N}(0,Q)} \left\{ \|H_W^T \eta\| \right\}}_{\substack{\text{stochastic} \\ \text{term}}} \\ \qquad \leq O(1)\mathrm{RiskB}[W]. \end{array}$$

$$\boxed{\begin{array}{l}\forall W \succeq 0 \exists H_W: \\ \underbrace{\mathbf{E}_{\xi\sim\mathcal{N}(0,W)}\left\{\|[B-H_W^TA]\xi\|\right\}}_{\text{bias}} + \underbrace{\mathbf{E}_{\eta\sim\mathcal{N}(0,Q)}\left\{\|H_W^T\eta\|\right\}}_{\substack{\text{stochastic}\\ \text{term}}} \\ \qquad \le O(1)\text{RiskB}[W].\end{array}} \qquad (!)$$

C. The key component of the proof is the fact that the efficiently computable upper bound on $\mathbf{E}_{\zeta\sim\mathcal{N}(0,Z)}\left\{\|U^T\zeta\|\right\}$ which we used when building good linear estimate is tight:

Lemma. *Let $\zeta \sim \mathcal{N}(0,Z)$ be zero mean N-dimensional Gaussian vector, U be a $N\times\nu$ matrix, and the unit ball $\mathcal{B}_*$ of the norm conjugate to $\|\cdot\|$ be an ellitope:*

$$\mathcal{B}_* = \{u : \exists r \in \mathcal{R}, y : u = My, y^TR_\ell y \le r_\ell, \ell \le L\}.$$

Then the efficiently computable upper bound

$$\Psi_Z(U) = \min_{G,\mu}\left\{\phi_{\mathcal{R}}(\mu) + \text{Tr}(ZG) : \mu \ge 0, \left[\begin{array}{c|c}\sum_\ell \mu_\ell R_\ell & \frac{1}{2}M^TU^T \\ \hline \frac{1}{2}UM & G\end{array}\right] \succeq 0\right\}$$

on $\mathbf{E}_{\zeta\sim\mathcal{N}(0,Z)}\left\{\|U^T\zeta\|\right\}$ is tight:

$$\Psi_Z(U) \le O(1)\sqrt{\ln(2L)}\mathbf{E}_{\zeta\sim\mathcal{N}(0,Z)}\left\{\|U^T\zeta\|\right\}.$$

Besides this, the bound is convex in U and concave in $Z \succeq 0$.

- Lemma combines with (!) to imply that

$$\begin{array}{l}\forall W \succeq 0: \\ \min_H\left\{\Psi_W(B^T - A^TH) + \Psi_Q(H)\right\} \le O(1)\sqrt{\ln(2L)}\text{RiskB}[W]\end{array}$$

$$\boxed{\begin{array}{l}\forall W \succeq 0:\\ \min\limits_H \left\{\Psi_W(B^T - A^T H) + \Psi_Q(H)\right\} \leq O(1)\sqrt{\ln(2L)}\mathrm{RiskB}[W]\end{array}}$$

D. For $0 < \rho \leq 1$, let

$$\begin{array}{rcl}\mathcal{Q}_\rho & = & \{W \succeq 0 : \exists t \in \mathcal{T} : \mathrm{Tr}(S_k W) \leq \rho t_k, k \leq K\} = \rho \mathcal{Q}_1,\\ \mathrm{Opt}(\rho) & = & \max\limits_{W \in \mathcal{Q}_\rho} \min\limits_H \left[\Psi_W(B^T - A^T H) + \Psi_Q(H)\right]\\ & \leq & O(1)\sqrt{\ln(2L)} \max\limits_W \{\mathrm{RiskB}[W] : W \in \mathcal{Q}_\rho\}\end{array}$$

It turns out that

D.1. By conic duality, $\mathrm{Opt} = \mathrm{Opt}(1)$

D.2. $\mathrm{Opt}(\rho) \geq \sqrt{\rho}\mathrm{Opt}(1)$, $0 \leq \rho \leq 1$

D.3. By the same argument as in the proof of tightness of the SDP upper bound on the maximum of a quadratic form over an ellitope, *when $W \in \mathcal{Q}_\rho$ and $\xi \sim \mathcal{N}(0, W)$, the probability for ξ to take value outside of $\mathcal{X}$ rapidly goes to 0 as $\rho \to +0$:*

$$\forall (\rho \leq 1, W \in \mathcal{Q}_\rho) : \mathrm{Prob}_{\xi \sim \mathcal{N}(0,W)} \{\xi \notin \mathcal{X}\} \leq O(1) K \exp\{-O(1)/\rho\}.$$

By **D.3**, for properly selected “moderately small” ρ one has

$$\max_W \{\mathrm{RiskB}[W] : W \in \mathcal{Q}_\rho\} \leq O(1)\mathrm{RiskOpt}_{\|\cdot\|,Q}[\mathcal{X}]$$

$\Rightarrow$ [by **D.1-2**] *For “moderately small” ρ one has*

$$\mathrm{Opt} \leq O(1)\sqrt{\ln(2L)/\rho}\,\mathrm{RiskOpt}_{\|\cdot\|,Q}[\mathcal{X}]. \qquad (\#)$$

Simple computation shows that with properly selected “moderately small” ρ, $(\#)$ implies the announced in Theorem upper bound on Opt.

From Ellitopes to Spectratopes

♠ **Fact:** *All our results extend from ellitopes – sets of the form*

$$\begin{array}{c}\mathcal{Y}=\{y\in\mathbb{R}^N:\exists t\in\mathcal{T},z:y=Pz,z^TS_kz\leq t_k,k\leq K\}\\ \left[\begin{array}{c}S_k\succeq 0,\sum_k S_k\succ 0\\ \mathcal{T}\subset\mathbb{R}^K_+: \text{monotone convex compact intersecting int}\,\mathbb{R}^K_+\end{array}\right]\end{array}\qquad (E)$$

which played the roles of signal sets, ranges of bounded noise, and the unit balls of the norms conjugate to $\|\cdot\|$, *to a wider family – spectratopes*

$$\begin{array}{c}\mathcal{Y}=\{y\in\mathbb{R}^N:\exists t\in\mathcal{T},z:y=Pz,S_k^2[z]\preceq t_kI_{d_k},k\leq K\}\\ \left[\begin{array}{c}S_k[z]=\sum_j z_jS^{kj},S^{kj}\in\mathbf{S}^{d_k},z\neq 0\Rightarrow\sum_k S_k^2[z]\neq 0\\ \mathcal{T}\text{ as in }(E)\end{array}\right]\end{array}\qquad (S)$$

With this extension, we get, e.g., access to

• matrix boxes $\mathcal{X}=\{x\in\mathbb{R}^{p\times q}:\|x\|_{2,2}\leq 1\}$ or their symmetric versions $\mathcal{X}=\{x\in\mathbf{S}^p_+:-I\preceq x\preceq I\}$ as signal sets

• nuclear norm $\|u\|_{\rm nuc}$ (sum of singular values of a matrix) as the norm quantifying recovery error

♠ **Modifications** of the results when passing from ellitopes to spectratopes are as follows:

A. *The "size" K of an ellitope (E)* (logs of these sizes participate in our tightness factors) *in the case of spectratope (S) becomes $D=\sum_k d_k$*

B. SDP relaxation bound for the quantity

$$\begin{array}{rcl}\mathrm{Opt}_* & = & \max\limits_{y}\left\{y^TBy : \exists t\in\mathcal{T}, z : y = Pz, S_k^2[z]\preceq t_kI_{d_k}, k\leq K\right\}\\ & = & \max\limits_{z,t}\left\{z^T\widehat{B}z : t\in\mathcal{T}, S_k^2[z]\preceq t_kI_{d_k}, k\leq K\right\}, \widehat{B} = P^TBP\end{array}$$

is as follows:

We associate with $S_k[z]=\sum_j z_jS^{kj}$, $S^{kj}\in\mathbf{S}^{d_k}$, two linear mappings:

$$\begin{array}{l} Q\mapsto \mathcal{S}_k[Q] : \mathbf{S}^{\dim z}\to\mathbf{S}^{d_k} : \\ \qquad \mathcal{S}_k[Q]=\sum_{i,j}\frac{1}{2}Q_{ij}[S^{ki}S^{kj}+S^{kj}S^{ki}]\\ \Lambda\mapsto\mathcal{S}_k^*[\Lambda] : \mathbf{S}^{d_k}\to\mathbf{S}^{\dim z} : \\ \qquad \left[\mathcal{S}_k^*[\Lambda]\right]_{ij}=\frac{1}{2}\mathrm{Tr}(\Lambda[S^{ki}S^{kj}+S^{kj}S^{ki}])\end{array}$$

Note:

- $S_k^2[z]=\mathcal{S}_k[zz^T]$
- *the mappings $\mathcal{S}_k$ and $\mathcal{S}_k^*$ are conjugates of each other w.r.t. to the Frobenius inner product:*

$$\mathrm{Tr}(\mathcal{S}_k[Q]\Lambda)=\mathrm{Tr}(Q\mathcal{S}_k^*[\Lambda])\ \forall(Q\in\mathbf{S}^{\dim z},\Lambda\in\mathbf{S}^{d_k})$$

Selecting $\Lambda_k\succeq 0$, $k\leq K$, such that $\sum_k\mathcal{S}_k^*[\Lambda_K]\succeq\widehat{B}$, for

$$z\in\mathcal{Z}=\{z : \exists t\in\mathcal{T} : S_k^2[z]\preceq t_kI_{d_k}, k\leq K\}$$

we have $\exists t\in\mathcal{T} : S_k^2[z]\preceq t_kI_{d_k}$, $k\leq K\Rightarrow$

$$\begin{array}{l} z^T\widehat{B}z\leq z^T\left[\sum_k\mathcal{S}_k^*[\Lambda_k]\right]z=\sum_k z^T\mathcal{S}_k^*[\Lambda_k]z=\sum_k\mathrm{Tr}(\mathcal{S}_k^*[\Lambda_k][zz^T])\\ =\sum_k\mathrm{Tr}(\Lambda_k\mathcal{S}_k[zz^T])=\sum_k\mathrm{Tr}(\Lambda_kS_k^2[z])\leq\sum_k t_k\mathrm{Tr}(\Lambda_k)\leq\phi_{\mathcal{T}}(\lambda[\Lambda]),\\ \qquad \phi_{\mathcal{T}}(\lambda)=\max\limits_{t\in\mathcal{T}}t^T\lambda,\ \lambda[\Lambda]=[\mathrm{Tr}(\Lambda_1);...;\mathrm{Tr}(\Lambda_K)]\end{array}$$

$\Rightarrow$

$$\mathrm{Opt}_*\leq\mathrm{Opt}:=\min_{\Lambda=\{\Lambda_k,k\leq K\}}\left\{\phi_{\mathcal{T}}(\lambda[\Lambda]) : \Lambda_k\succeq 0, k\leq K, \widehat{B}\preceq\sum_k\mathcal{S}_k^*[\Lambda_k]\right\}$$

♠ **Theorem** [Ju&N,'17] *SDP relaxation bound*

$$\mathrm{Opt} := \min_{\Lambda=\{\Lambda_k, k\le K\}} \left\{ \phi_{\mathcal{T}}(\lambda[\Lambda]) : \Lambda_k \succeq 0, k \le K, \widehat{B} \preceq \sum_k \mathcal{S}_k^*[\Lambda_k] \right\}$$

on the quantity

$$\begin{array}{rcl}\mathrm{Opt}_* & = & \max_y \left\{ y^T B y : \exists t \in \mathcal{T}, z : y = Pz, S_k^2[z] \preceq t_k I_{d_k}, k \le K \right\} \\ & = & \max_{z,t} \left\{ z^T \widehat{B} z : t \in \mathcal{T}, S_k^2[z] \preceq t_k I_{d_k}, k \le K \right\}\end{array}$$

is tight:

$$\mathrm{Opt}_* \le \mathrm{Opt} \le 2\ln(2\sum_k d_k)\mathrm{Opt}_*.$$

Note: The role of elementary Mini-Lemma in the spectratopic case is played by the following fundamental matrix concentration result:
Noncommutative Khintchine Inequality [Lust-Picard 1986, Pisier 1998, Buchholz 2001] *Let $A_i \in \mathbf{S}^d$, $1 \le i \le N$, be deterministic matrices such that*

$$\sum_i A_i^2 \preceq I_d,$$

and let ζ be N-dimensional Rademacher random vector. Then for all $s \ge 0$ it holds

$$\mathrm{Prob}\left\{ \|\sum_i \zeta_i A_i\|_{2,2} \ge s \right\} \le 2d \exp\{-s^2/2\}.$$