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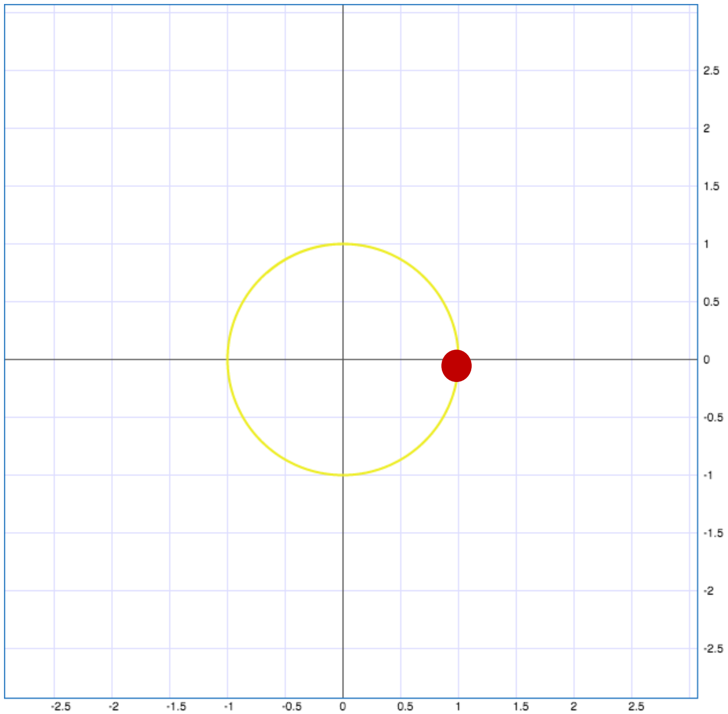
Four stories on the stability of linear systems

Preface

Consider a linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \cos t \end{pmatrix} \qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$x(0) = 1, \qquad y(0) = 0$$



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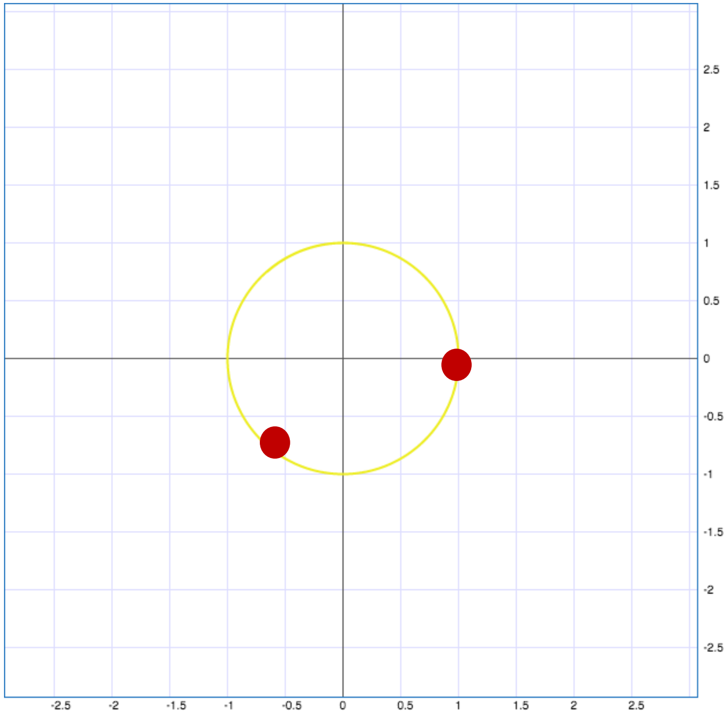
$x(0) = 1 + \varepsilon,\qquad y(0) = -\varepsilon \qquad\qquad \varepsilon = 10^{-20}$

$t = 10$ sec.



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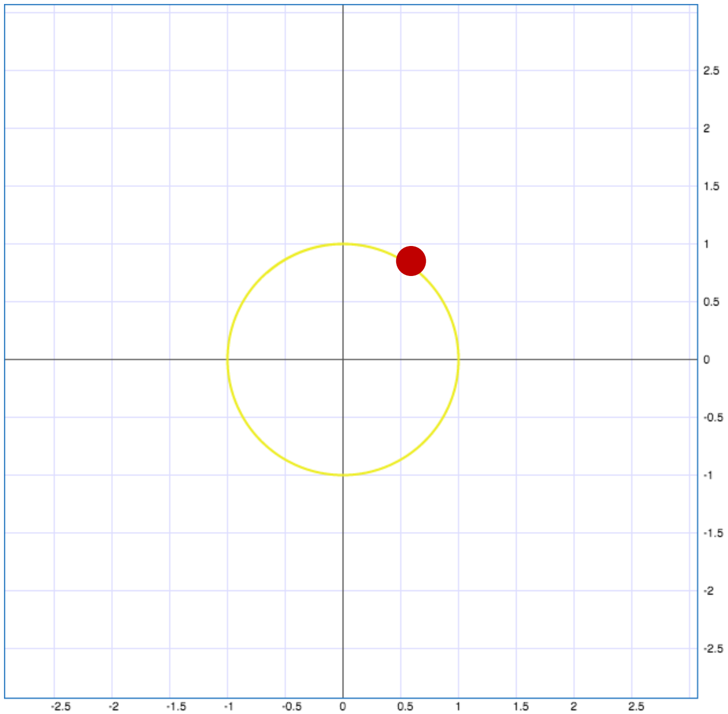
$t = 20$ sec.



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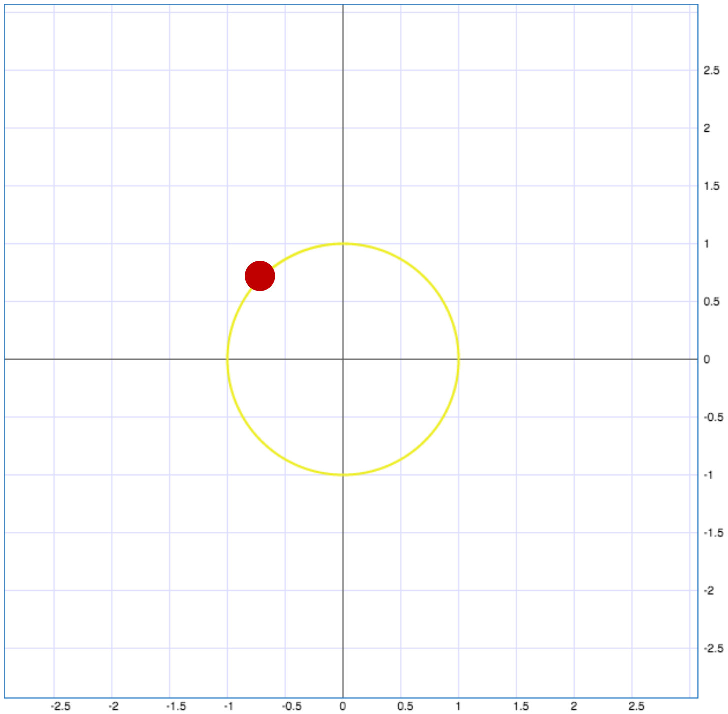
$x(0) = 1 + \varepsilon, \qquad y(0) = -\varepsilon \qquad\qquad \varepsilon = 10^{-20}$

$t = 40$ sec.



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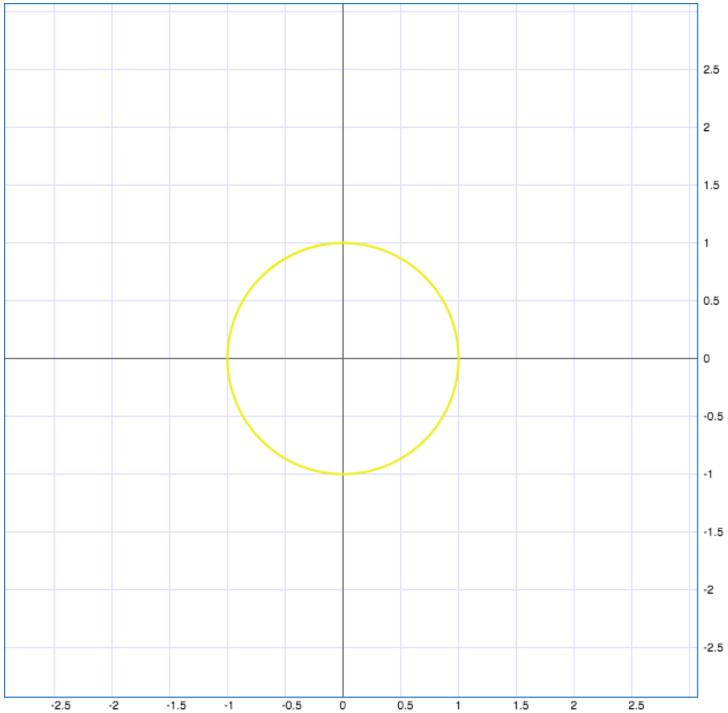
$x(0) = 1 + \varepsilon,\quad y(0) = -\varepsilon \qquad \varepsilon = 10^{-20}$

$t = 50$ sec.



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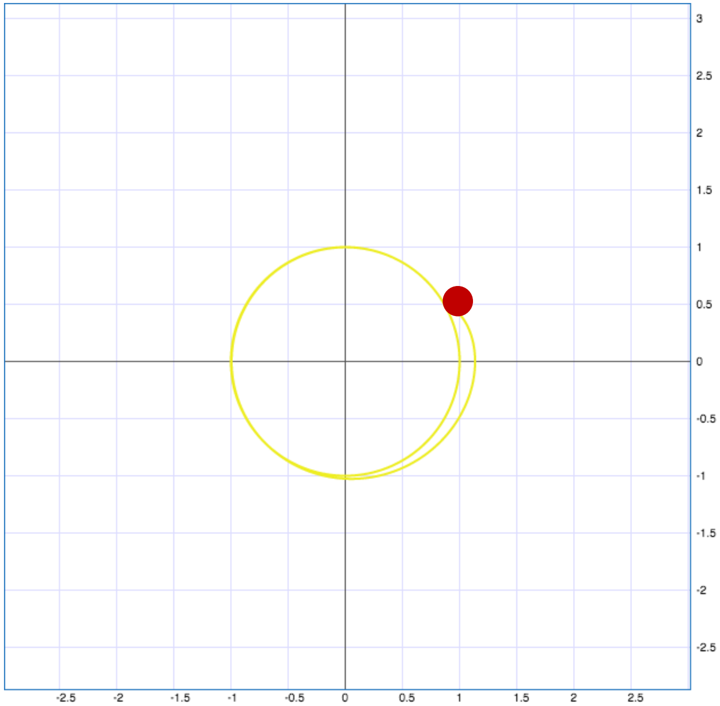
$x(0) = 1 + \varepsilon, \qquad y(0) = -\varepsilon \qquad\qquad \varepsilon = 10^{-20}$

$t = 45$ sec.

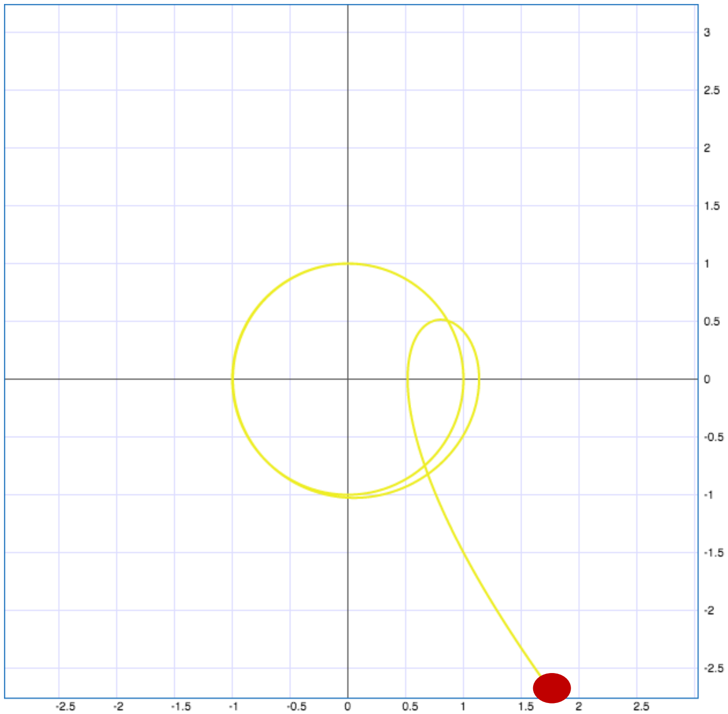


Consider a linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

$x(0) = 1 + \varepsilon, \quad y(0) = -\varepsilon \qquad \varepsilon = 10^{-20}$

$t = 48$ sec.



In $t = 60$ sec. the point will be in 14 km. from the center.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad \lambda_1 = \lambda_2 = 1 \qquad \Rightarrow \quad \text{the system is unstable}$$

A linear dynamical system with continuous time:

$$\dot{x}(t) = A\,x(t), \quad t \in [0,+\infty), \quad x(0) = x_0$$

A system is stable if all trajectories tend to zero
(Hurwitz stability)

$$\operatorname{Re}\lambda_k < 0\ , \quad \lambda_k \in sp(A)$$

A linear dynamical system with discrete time:

$$x_{k+1} = A\,x_k\,, \quad k \in \mathbb{Z}_+ \quad x_0 \text{ is given}$$

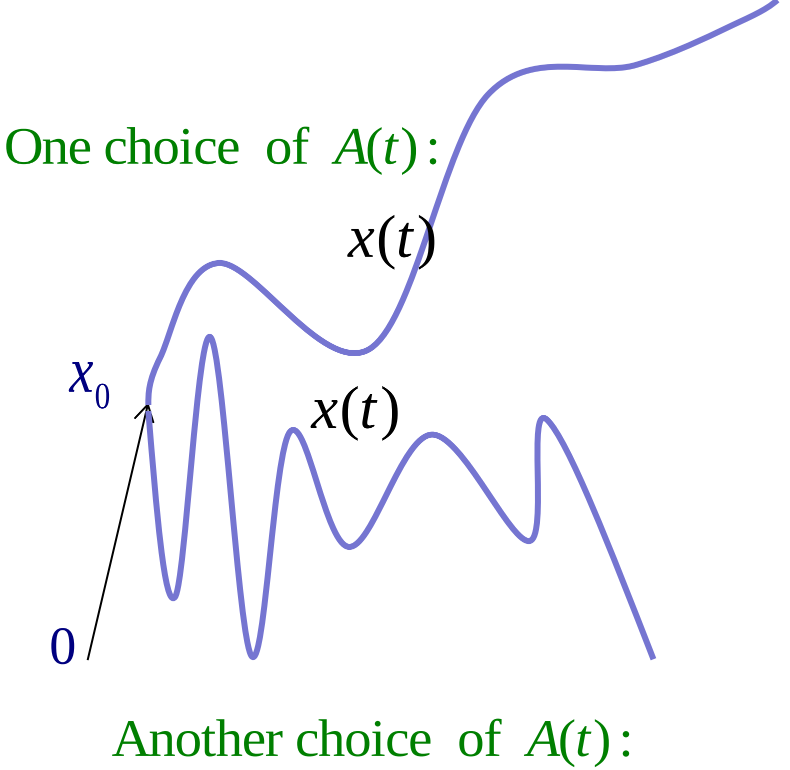
A system is stable if all trajectories tend to zero
(Schur stability)

$$|\,\lambda_k\,| < 1\ , \quad \lambda_k \in sp(A)$$

Linear switching systems

E.Pyatnitsky, V.Opoytsev, A.Molchanov (1980),
N.Barabanov, V.Kozyakin (1988), L.Gurvits (1996)

P.Mason, M.Sigalotti, M.Margaliot, F.Blanchini, S.Miani,
U.Boskian, D.Liberzon, and many others



Consider a system of linear ODE

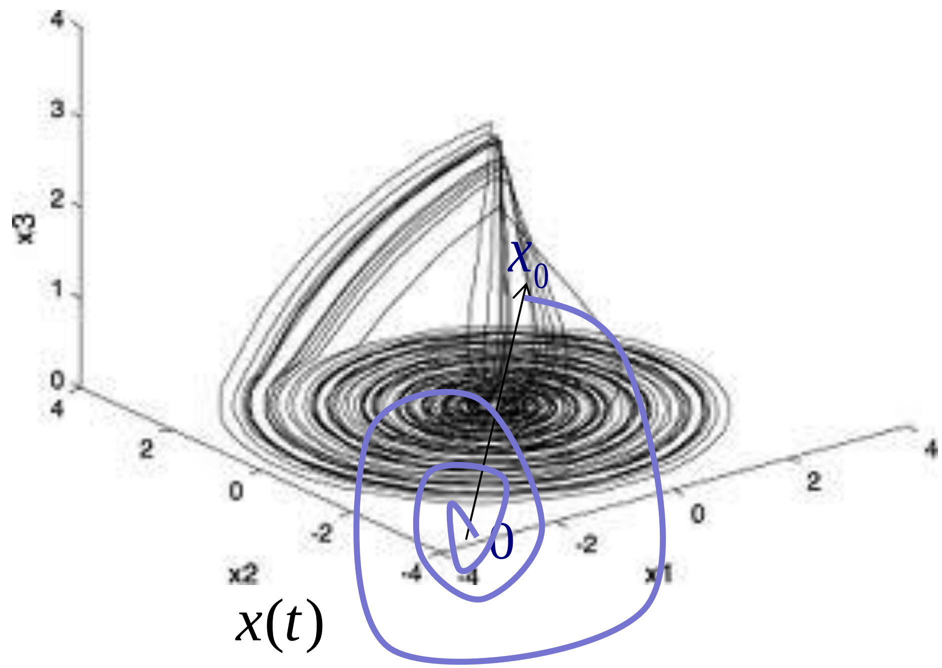
$$\dot{x}(t) = A(t)x(t), \quad t \in [0, +\infty)$$
$$x(0) = x_0$$

$x = \left(x_1(t), \dots, x_d(t) \right)$, $A(t)$ is a $d \times d$ - matrix,

$\forall\, t \in [0, +\infty) \quad A(t) \in U$

U is a compact set of matrices

Example. $U = \{A_1, A_2\}$



Definition 1. The system is stable if $x(t) \to 0$ as $t \to \infty$ for every x_0 and for every choice of $A(t)$.

Story 1

Linear switching systems. How to find the optimal Lyapunov function?

Literature

There is a wide literature on switched systems.

A couple of monographs

- D. Liberzon: Switching in systems and control. Systems & Control: Foundations & Applications. Birkhäuser, 2003.
- Handbook of hybrid systems control. Theory, tools, applications. Edited by J. Lunze and F. Lamnabhi - Lagarrigue. Cambridge University Press, 2009.

How to decide the stability ?

$$\dot{x}(t) = A(t)x(t), \quad t \in [0,+\infty)$$
$$x(0) = x_0$$

$x = \ x_1(t),..., x_d(t)\ $, $A(t)$ is a measurable control function,

$A(t) \in U$ for almost all $t \in [0,+\infty)$

If U consists of one matrix A, then $x(t) = e^{tA} x_0$

the system is stable $\Leftrightarrow$ Re $(\lambda) < 0$, for all eigenvalues λ of A

(i.e., A is a Hurwitz stable matrix)

What to do if Card (U) $\geq$ 2 ?

Necessary condition:

if the system is stable, then all matrices from co (U) are Hurwitz stable.

Not sufficient already for d =2 (L.Gurvits, 1999)

Conjecture 1 (P.Mason, R.Shorten, 2003) Sufficient for positive systems.

A system is positive if all $A \in U$ is Motzler, i.e. $A_{ij} \geq 0$, $i \neq j$.

A matrix is Motzler $\Leftrightarrow$ $e^{tA} \geq 0$ for all t.

Example.

$$A = \begin{pmatrix} -100 & 1 & 2 \\ 0 & -15 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

The conjecture is proved for $d = 2$ by P.Mason and R.Shorten (2003)

and disproved for d $\geq$ 3 by L.Faishil, M.Margaliot and P.Chigansky (2011)

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -10 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -10 & 0 & 10 \\ 0 & -10 & 0 \\ 0 & 10 & -1 \end{pmatrix}.$$

The optimal control approach

E.Pyatnitsky, V.Opoytsev, A.Molchanov, L.Rapoport (1970s - 1980s)

Idea: to find the worst (fastest growing) trajectory solving the optimal control problem by Pontryagin's maximum principle:

$$\|x(T)\| \to \max$$

$$\dot{x}(t) = A(t)x(t), \quad t \in [0,T]$$

$$x(0) = x_0$$

$$A(t) \in U \text{ for almost all } t \in [0,T]$$

Solved usually numerically. The problem is difficult.

For positive systems it is easier (M.Margaliot, 2013), but still for small *d*.

The Lyapunov function

Definition. A continuous function $f : \mathbb{R}^d \to \mathbb{R}_+$ is called Lyapunov function if

1) $f(x) > 0, \; x \neq 0,$

2) $f(\alpha x) = \alpha f(x)\,, \quad \alpha \geq 0,$

3) $f(x(t))$ is decreasing in t, for everty trajectory $x(t)$ of the system.

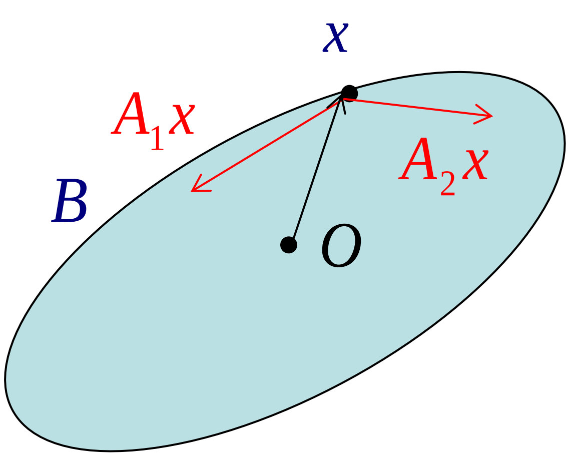
There is a Lyapunov function $\Rightarrow$ the system is stable.

The system is stable $\Rightarrow$ there is a convex symmetric Lyapunov function (norm).

(L.Opoitsev (1977), A.Molchanov, E.Pyatnitsky (1980), N.Barabanov (1989)).

The Lyapunov norm

Take a unit ball of that norm: $B = \{ x \in \mathbb{R}^d \mid f(x) \leq 1 \}$



A norm $f(x)$ is a Lyapunov norm

$\Updownarrow$

for every $x \in \partial B$ and for every $A \in U$

the vector Ax starting at the point x is "directed inside" B.

A quadratic Lyapunov function

Thus, to prove the stability it suffices to present a Lyapunov function $f(x)$.

How to find $f(x)$?

This is equivalent to constucting a convex body B.

The most natural choice is a quadratic function f(x) = $\sqrt{x^T M x}$, where M is p.s.d. matrix.

A matrix M $\succ 0$ defines a Lyapunov function $\Leftrightarrow$ $A^T M + M A \prec 0 \quad \forall A \in U$

This is an s.d.p. problem, it can be efficiently solved.

However, this is just a sufficient condition. In practice, it is far from being necessary.

Very often a quadratic Lyapunov function does not exists, although the system is stable.

There are other types of Lyapunov functions in the literature
(piecewise-quadratic, polyhedral, sum-of-squares, etc.)

Definition. The Lyapunov exponent $\sigma(A)$ is the infimum of numbers α such that
$\|x(t)\| \leq C\, e^{\alpha t}$ for all trajectories $x(t)$.

The system is stable if and only if $\sigma(A) < 0$.

Theorem (N.Barabanov, 1989). For an arbitrary irreducible system there exists an invariant Lyapunov norm $f(x) = \|x\|$, for which two conditions are satisfied:

1) $\|x(t)\| \leq \|x(0)\| e^{\sigma t}$ for all trajectories $x(t)$.

2) There is a trajectory $x(t)$ such that $\|x(t)\| = \|x(0)\| e^{\sigma t}$ for all t.

In case $\sigma = 0$

(The geometric interpretation). There is a symmetric about the origin convex body $G \subset \mathbb{R}^d$ such that all tracterures started in G never leave it, and there is at least one trajectory that entirely lies on the boundary of G.

The invariant norm may not be well-approximated by quadratic functions

Polytope (piecewise-linear) Lyapunov function :

$$f(x) = \max_{i=1,\dots,N} (a_i , x)$$

Theorem (F. Blanchini, S. Miani, 1996) For any stable LSS there exists a polytope Lyapunov norm.

The polytope norm is extremely difficult to compute already in the dimension 3

To construct a polytope norm we consider first the discrete systems

$$x_{k+1} = A x_k, \quad k \in \mathbb{Z}_+ \quad x_0 \text{ is given}$$

A system is stable if all trajectories tend to zero (Schur stability)

Theorem (N.Barabanov, 1988) A discrete system is stable if and only if its joint spectral radius is smaller than one.

The Joint spectral radius (JSR)

$A_1, \cdots A_m$ are linear operators in $\mathbb{R}^d$

$$\hat{\rho}(A_1, \cdots, A_m) = \lim_{k\to\infty} \max_{d_1,\dots,d_k \in \{1,\dots,m\}} \left\| A_{d_1} \cdots A_{d_k} \right\|^{1/k}$$

J.C.Rota, G.Strang (1960) -- Normed algebras

N.Barabanov, V.Kozyakin, E.Pyatnitsky, V.Opoytsev, L.Gurvits, …(1988)

Linear switching systems

C.Micchelli, H.Prautzch, W.Dahmen, A.Levin, N.Dyn, P.Oswald,…… (1989)

Subdivision algorithms

I.Daubechies, J.Lagarias , C.Heil, D.Strang, … (1991)

Wavelets

The Joint spectral radius (JSR)

$A_1, \cdots, A_m$ are linear operators in $\mathbb{R}^d$

$$\hat{\rho}(A_1 \cdots, A_m) = \lim_{k\to\infty} \max_{d_1,\dots,d_k \in \{1,\dots,m\}} \left\| A_{d_1} \cdots A_{d_k} \right\|^{1/k}$$

Example 1. If m = 1, we have a family of one matrix {A};

$$\text{then} \quad \hat{\rho}(A) = \lim_{k\to\infty} \left\| A^k \right\|^{1/k} = \max_{j=1,\dots,d} |\lambda_j|$$

Example 2. If all the matrices $A_1, \dots, A_m$ are orthogonal, then $\left\| A_{d_1} \cdots A_{d_k} \right\| = 1$,

hence $\hat{\rho} = 1$

Example 3. If all the matrices $A_1, \dots, A_m$ are diagonal, then

$$\boxed{\hat{\rho} = \max\{\rho(A_1), \dots, \rho(A_m)\}}$$

The same is true if all the matrices

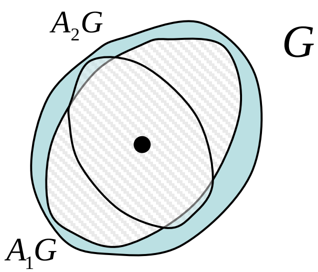
- commute
- are upper (lower) triangular
- are symmetric

$$\boxed{\text{In general, however,} \quad \hat{\rho} > \max\{\rho(A_1), \dots, \rho(A_m)\}}$$

The Joint spectral radius (JSR)

$A_1, \cdots, A_m$ are linear operators in $\mathbb{R}^d$

$$\hat{\rho}(A_1, \cdots, A_m) = \lim_{k\to\infty} \max_{d_1,\dots,d_k \in \{1,\dots,m\}} \left\| A_{d_1} \cdots A_{d_k} \right\|^{1/k}$$



The geometric sense:

$\hat{\rho} < 1 \Leftrightarrow$ there exists a norm $\|\bullet\|$ in $\mathbb{R}^d$ such that $\|A_i\| < 1$ for all $i = 1, \dots, m$

Taking the unit ball in that norm:

$\hat{\rho} < 1 \Leftrightarrow$ there exists a symmetric convex body $G \subset \mathbb{R}^d$ such that $A_i G \subset \operatorname{int} G, \quad i = 1, \dots, m$

Example 4. If all the matrices $A_1, \dots, A_m$ are symmetric, then one can take G a Euclidean ball $\Rightarrow \hat{\rho} = \max\{\rho(A_1), \dots, \rho(A_m)\}$

Example 5. If all $A_1, \dots, A_m$ are orthogonal projections, then one can take the same Euclidean ball $\Rightarrow \hat{\rho} = 1$

Other applications of the Joint Spectral Radius

- Probability
- Combinatorics
- Number theory
- Mathematical economics
- Discrete math

How to compute or estimate ?

Blondel, Tsitsiklis (1997-2000).

- **The problem of JSR computing for nonnegative rational matrices in NP-hard**

- **The problem, whether JSR is less than 1 (for rational matrices) is algorithmically undecidable whenever d > 46.**

- **There is no polynomial-time algorithm, with respect to both the dimension d and the accuracy**

The main inequality for JSR

For every k we have

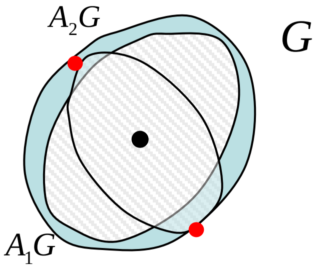
$$\max_{d_1,\ldots,d_k \in \{1,\ldots,m\}} \left[\rho(A_{d_1} \cdots A_{d_k}) \right]^{1/k} \le \hat{\rho} \le \max_{d_1,\ldots,d_k \in \{1,\ldots,m\}} \left\| A_{d_1} \cdots A_{d_k} \right\|^{1/k}$$

The concept of extremal norm

Definition. A norm $\| . \|$ is called extremal if $\| A_j \| \le \hat{\rho}\,, \quad j = 1, \ldots, m$.

For the extremal norm the convergence is realized within one step.

If G is the unit ball in that norm, then $A_j G \subset \hat{\rho}\, G\,, \quad j = 1, \ldots, m.$



Methods for estimating JSR

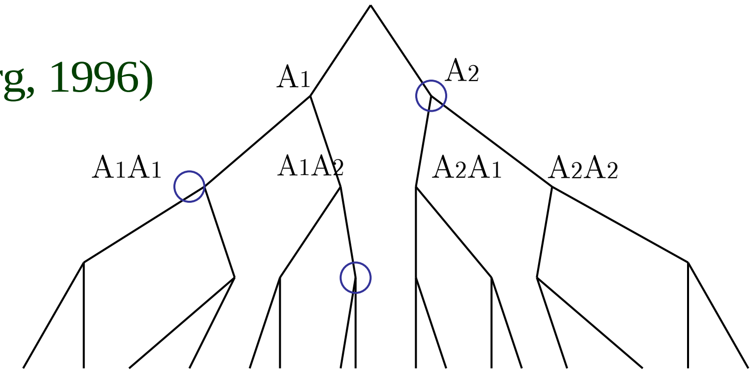
- ``By definition" (Daubechies, Lagarias, Heil, Strang, ... 1991)

Using the inequality

$$\max_{d_1,\ldots,d_k \in \{1,\ldots,m\}} \left[\rho(A_{d_1}\cdots A_{d_k}) \right]^{1/k} \le \hat{\rho} \le \max_{d_1,\ldots,d_k \in \{1,\ldots,m\}} \left\| A_{d_1}\cdots A_{d_k} \right\|^{1/k}$$

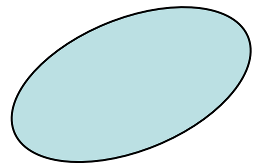
Very slow and rough. Only for small dimensions (d $\le$ 4)

- ``Branch and bound" algorithm (G.Grippenberg, 1996)



Pretty rough (relative error at least 2 - 5 percent), but much faster

- ``Best ellipsoidal norm''   (Ando, Shih (1998), Blondel, Nesterov, Theys (2004))



Approximates the extremal norm by ellipsoids, using SDP.

Works for high dimensions (up to d =20), but quite rough.

- ``Tensor products of matrices''   (P. (1997), Blondel, Nesterov (2005))

Approximates the extremal norm by even polynomials

Fast, but very rough

- ``Sum of squares algorithm''   (Parrilo, Jadbabaie (2008))

Approximates the extremal norm by some of squares polynomials.

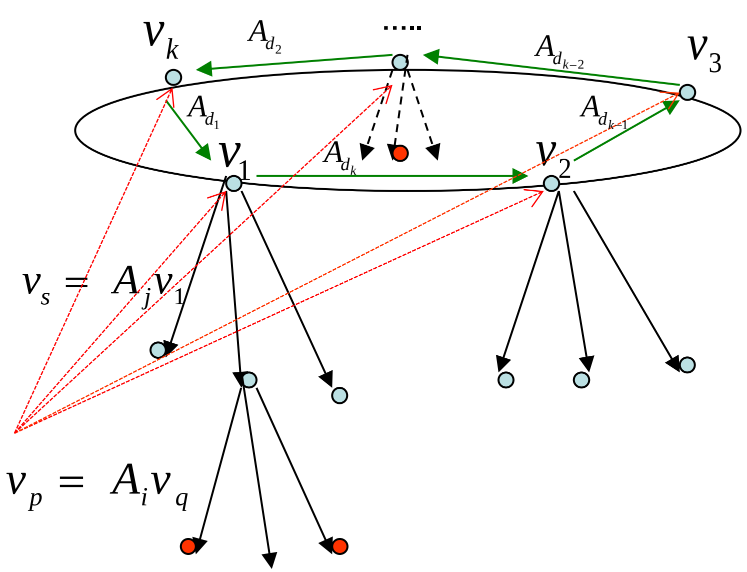
More or less the same complexity as the previous method.

The algorithm for exact JSR computation

N.Guglielmi, V.P., Found. Comput. Math, 13 (2013), 37-97

We take the leading eigenvector v_1 of $\Pi = A_{d_1} \cdots A_{d_k}$.

Set $v_j = A_{d_{k-j+2}} \cdots A_{d_k} v_1$, $j = 2, \dots, k$.



The ''dead'' branches

Every time we check if the new vertex is in the convex hull of the previous ones (this is a linear programming problem).

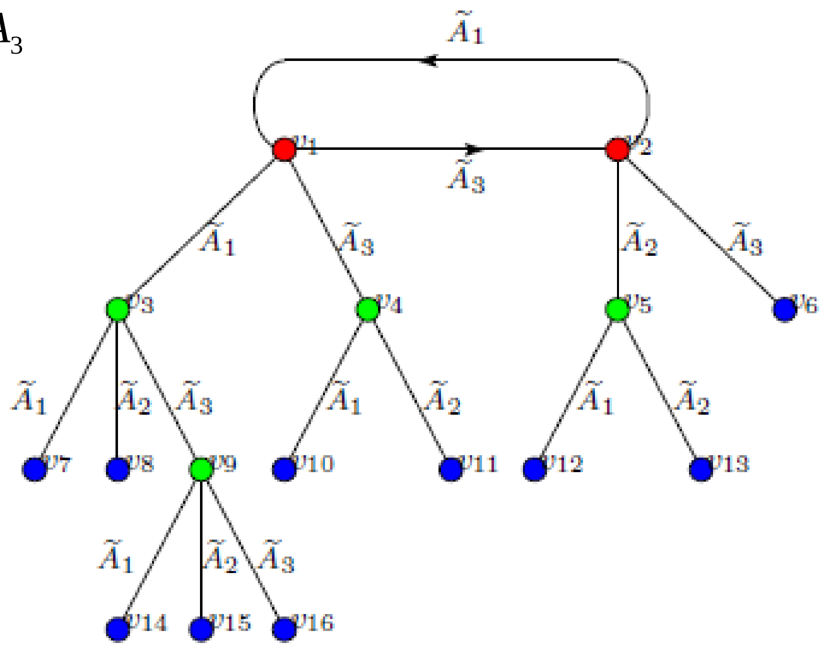
The algorithm terminates, when there are no new vertices.

The invariant polytope G is the convex hull of all vertices produced by the algorithm

The family of matrices from the problem of Euler partition function:

$$A_1 = \begin{pmatrix} 1&1&1&1&1&0&0\\ 0&1&1&1&1&0&0\\ 0&1&1&1&1&1&0\\ 0&1&1&1&1&1&0\\ 0&0&1&1&1&1&0\\ 0&0&1&1&1&1&1\\ 0&0&1&1&1&1&1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1&1&1&1&1&0&0\\ 1&1&1&1&1&0&0\\ 0&1&1&1&1&0&0\\ 0&1&1&1&1&1&0\\ 0&1&1&1&1&1&0\\ 0&0&1&1&1&1&0\\ 0&0&1&1&1&1&1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0\\ 1&1&1&1&1&0&0\\ 0&1&1&1&1&0&0\\ 0&1&1&1&1&1&0\\ 0&1&1&1&1&1&0\\ 0&0&1&1&1&1&0 \end{pmatrix}$$

We choose $\Pi = A_1 A_3$



$\boxed{\text{The extremal polytope } G_3 \text{ has 16 vertices.}}$

JSR computation for randomly generated matrices of dimensions d = 5, …, 20

	JSR				JSR		
d	# its	# vertices	s.m.p.	d	# its	# vertices	s.m.p.
5	3	14	A_1A_2	6	4	26	A_1
5	7	23	$A_1A_2^2$	6	9	51	A_1A_2
5	12	37	A_1	6	5	38	$A_1^2A_2$
7	17	100	A_1	8	19	117	$A_1^3A_2A_1^4A_2$
7	12	140	$A_1^3A_2A_1A_2$	8	8	49	A_1
7	24	223	$A_1^3A_2^2$	8	12	75	$A_1A_2^3$
9	18	177	$A_1^8A_2$	10	16	239	$A_1A_2^4$
9	13	172	$A_1^3A_2A_1A_2$	10	9	109	A_1
9	10	129	A_2	10	24	408	$(A_1^3A_2)^2A_2$
11	20	707	$A_1^3A_2^2$	12	31	1539	$A_1A_2A_1^2A_2^2$
11	14	340	$A_1^2A_2A_1A_2$	12	9	211	A_1A_2
11	12	183	$A_1^3A_2$	12	13	215	$A_1A_2^3$
15	18	715	$A_1^2A_2A_1A_2^4$	20	21	1539	A_1A_2
15	14	570	$A_1^4A_2$	20	16	1219	$A_1A_2^2$
15	14	390	A_2	20	16	1247	$A_1^2A_2^2$

JSR computation for positive matrices of dimension d = 100.

	JSR			LSR		
density	# its	# vertices	s.m.p.	# its	# vertices	s.l.p.
0.2	6	24	$A_1^2 A_2$	6	31	$A_1 A_2^2$
0.2	6	23	$A_1 A_2$	6	28	$A_1^2 A_2^2$
0.2	7	27	$A_1 A_2^3$	6	20	$A_1 A_2$
0.2	5	21	$A_1 A_2^2$	7	24	$A_1^2 A_2$
0.5	5	10	$A_1 A_2$	5	15	$A_1 A_2^2$
0.5	6	17	$A_1^2 A_2$	4	8	$A_1 A_2$
0.5	6	18	$A_1^2 A_2^2$	5	16	$A_1^2 A_2$
0.5	6	22	$A_1 A_2^3$	4 (6)	9 (14)	A_1 and A_2
0.8	4	7	$A_1 A_2$	4	7	$A_1 A_2$
0.8	7	18	$A_1^2 A_2$	6	14	$A_1^2 A_2^2$
0.8	5	14	$A_1 A_2^2$	9 (7)	14 (16)	A_1 and A_2
0.8	5	12	$A_1^3 A_2$	5	12	$A_1 A_2^2$

Conditions for finite terminating of the algorithm

Definition. A product $\Pi = A_{d_k} \cdots A_{d_1}$ is called dominant if $\rho(\Pi)=1$, and there is $q<1$ such that $\rho(\Delta) < q$ for any product $\Delta = A_{d_n} \cdots A_{d_1}$ that is not a power of Π nor of its cyclic permutations.

dominant $\not\Leftrightarrow$ **s.m.p.**

Theorem 1. The algorithm terminates within finite time if and only if the product Π is dominant.

The idea: to discretize the system, to construct an invariant politope for it and then to use it as a Lyapunov norm.

N.Guglielmi, L.Laglia, V.Protasov, *Found. Comput. Math. 17 (2017), 567-623.*

The upper and lower bounds for the Lyapunov exponent:

Fix $\tau > 0$, consider a discrete system $e^{\tau A}$ with matrices $e^{\tau A_k}$.

Let P be the invariant polytope for the family $e^{\tau A_1}, \dots, e^{\tau A_m}$

Define the two folloving values:

$$\beta(\tau) = \tau^{-1} \hat{\rho}(e^{\tau A_1}, \dots, e^{\tau A_m})$$

$$\alpha(\tau) = \inf \left\{ \alpha > 0 \mid \text{for each vertex } v \text{ of } P, \text{ for each } k, \text{ the vector } (A_k - \alpha I)v \text{ is directed inside P} \right\}$$

Theorem. For every family $A = \{A_1, \dots, A_m\}$ and for every $\tau > 0$, we have

$$\beta(\tau) \leq \sigma \leq \alpha(\tau)$$

Moreover, there is a constant $C > 0$ such that

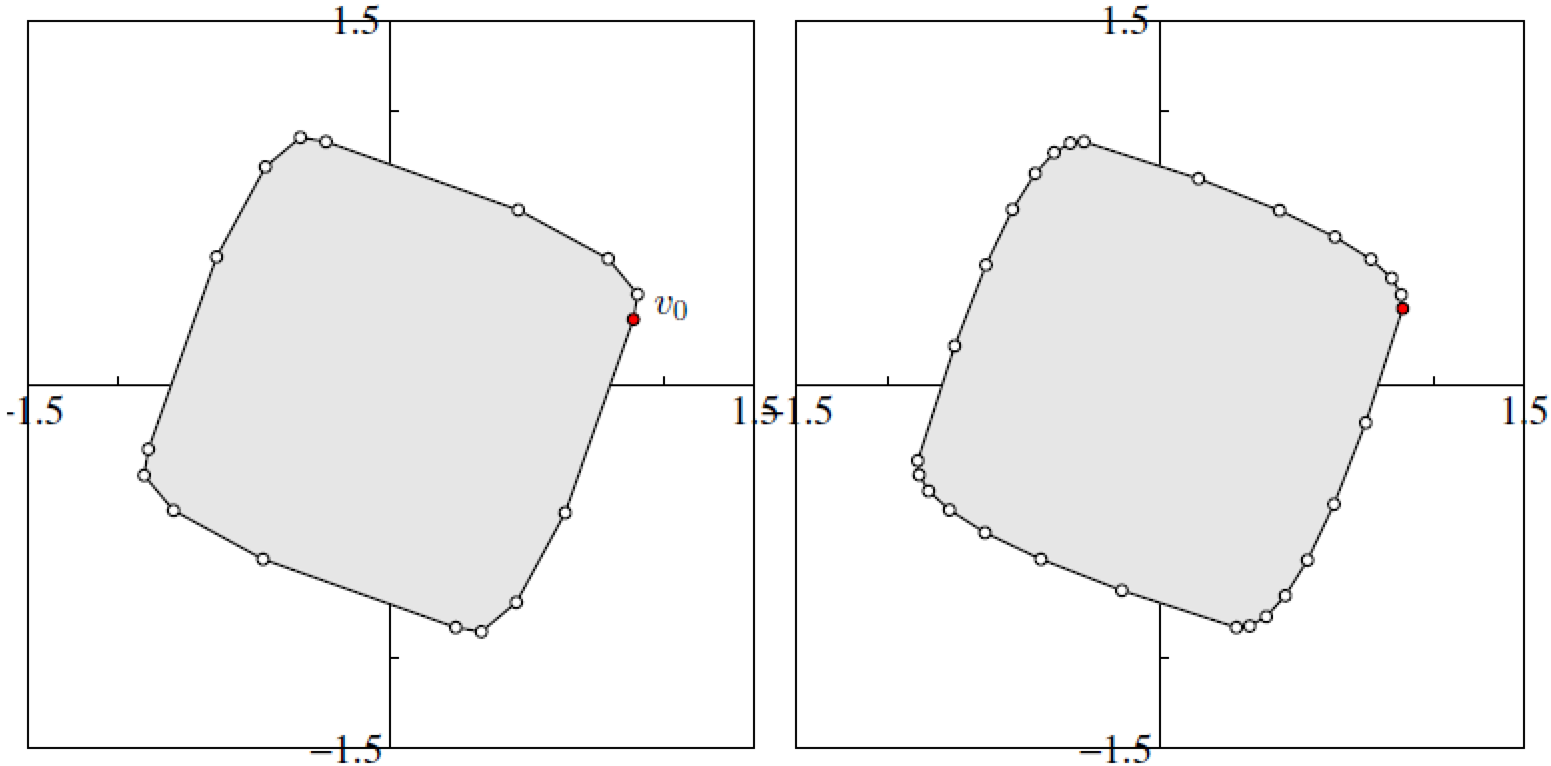
$$\alpha(\tau) - \beta(\tau) \leq C\tau, \qquad \tau \in \mathbb{R}_+$$

Constructing a polytope Lyapunov function for LSS.

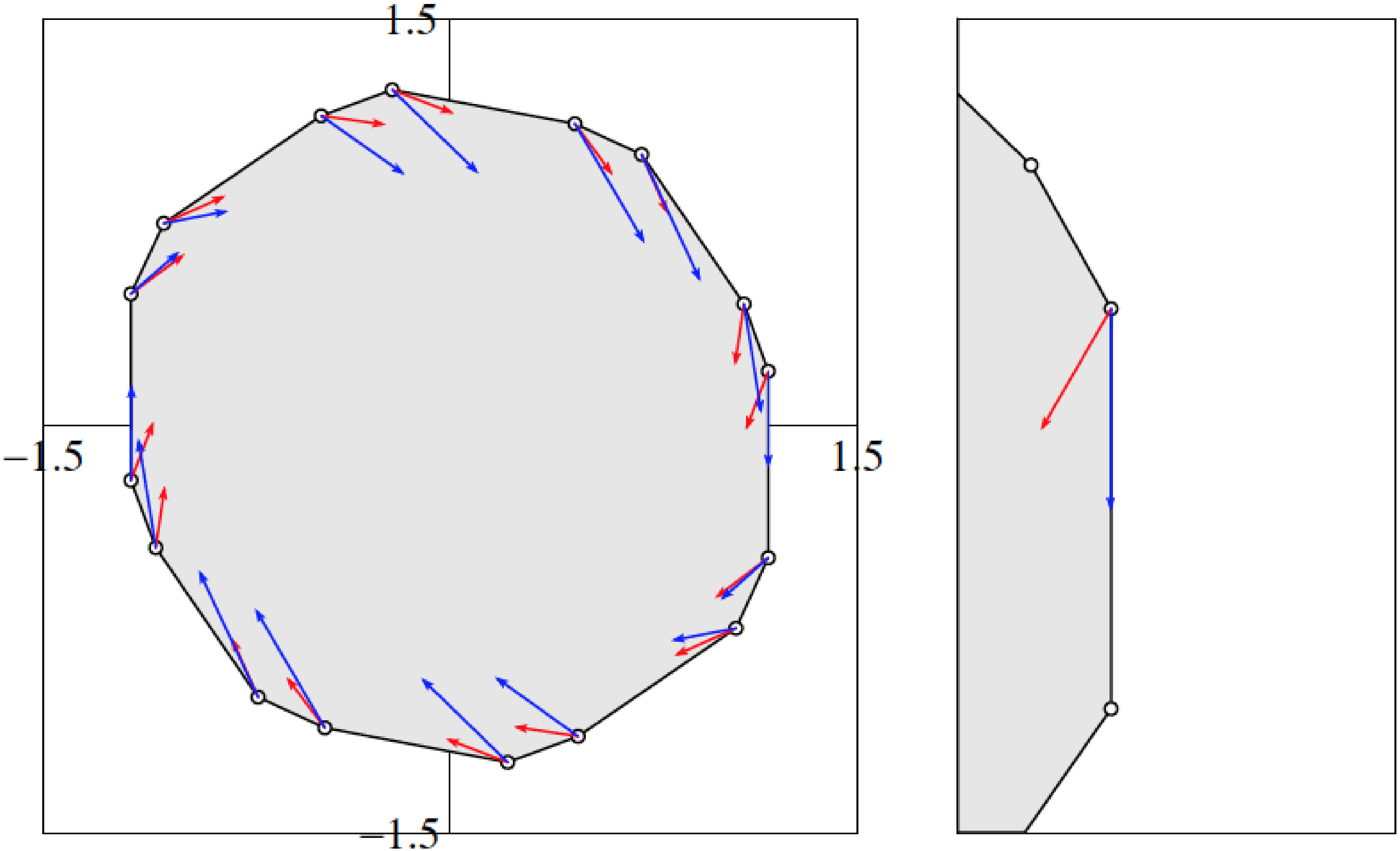
We take $\tau = \Delta t = 1$ and then divide it by 2 as many times as we can.

For each $\tau = \Delta t$ we find a dominating product of the family $I + \tau U$

and build an invariant polytope. This gives a lower and upper bounds for $\sigma(U)$.



$\Delta t = 1$ $\quad\quad$ $\Delta t = 1/2.$

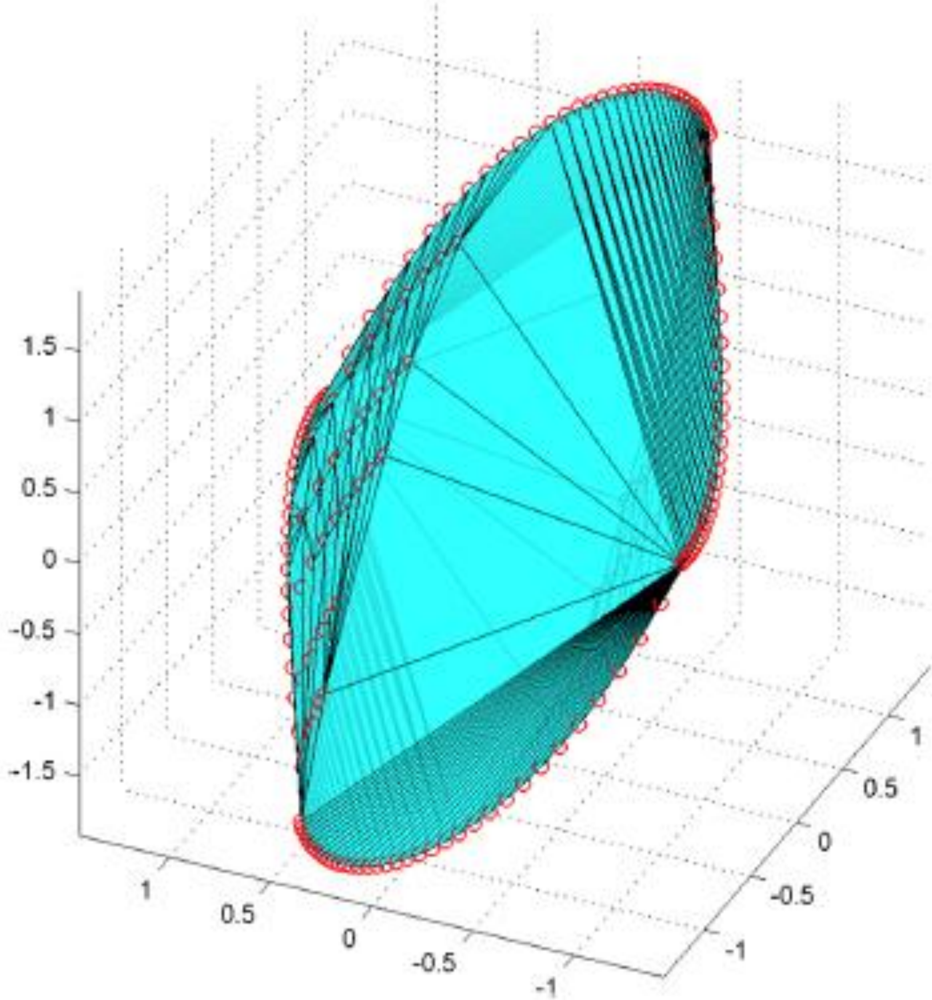


1.5
−1.5
1.5
−1.5

$$A_1 = \begin{pmatrix} -0.0822 & 0.0349 & -0.1182 \\ 0.0953 & -0.0897 & -0.1719 \\ 0.0787 & 0.0223 & -0.2781 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.1391 & 0.1397 & -0.0916 \\ 0.0338 & -0.1769 & -0.0707 \\ 0.7417 & 0.3028 & -0.4621 \end{pmatrix}.$$

Table 1 Approximation of the Lyapunov exponent (Example 1)

τ	β	α	γ	Π	ε	$\#V$
1/2	−0.0470	0.0074	0.0545	$B_1^{27} B_2^{29}$	0.05	163
1/2	−0.0470	−0.0148	0.0322	$B_1^{27} B_2^{29}$	0.025	322
1/4	−0.0410	0.0089	0.0550	$B_1^{55} B_2^{58}$	0.0125	423
1/4	−0.0410	−0.0243	0.0227	$B_1^{55} B_2^{58}$	0.005	1655



1.5
1
0.5
0
-0.5
-1
-1.5
1
0.5
0
-0.5
-1
1
0.5
0
-0.5
-1

For positive systems (i.e., defined by Metzler matrices) the method works much faster even in relatively high dimensions

Table 13 Statistics on LE computation for Metzler problems of dimension $d = 50$ with entries in $[-1, 1]$

τ	$L_{\min}$	$L_{\max}$	$\#V_{\min}$	$\#V_{\max}$	$\langle \#V \rangle$	$\gamma_{\min}$	$\gamma_{\max}$	$\langle \gamma \rangle$
1/4	1	2	2	2	2	0.321	0.624	0.5115
1/16	1	3	4	7	5	0.1798	0.334	0.2775
1/64	1	15	11	28	22	0.094	0.126	0.1143
1/128	1	15	23	74	55	0.079	0.097	0.0894

Story 2

The problem of the steplength in the Euler broken line and the Markov-Bernstein inequality for exponents.

Discretization of a linear switching system

We make the discretization wirh the stepsize $\tau > 0$

$$x_k = x(k\tau)\ ; \quad A_k = A(k\tau)\,, \quad k \in \mathbb{N}$$

$$\dot{x}(k\tau) \approx \frac{x(k\tau + \tau) - x(k\tau)}{\tau} = \frac{x_{k+1} - x_k}{\tau}$$

and obtain the discretized system:

$$x_{k+1} = (\mathrm{I} + \tau A_k) x_k\ , \quad k \in N$$

x_0 is given, $A_k \in U$

Theorem (A.Molchanov, E.Pyatnitsky, 1980).

If the discretization is stable for some $\tau_0 > 0$, then it is stable for all $\tau < \tau_0$ and the corresponding continuous system is stable.

How to decide the stability of the discretized system ?

$$x_{k+1} = (I + \tau A_k) x_k , \quad k \in N$$

$$x_0 \text{ is given, } A_k \in U$$

Denote $I + \tau A_k = B_k$. Then $x_{k+1} = B_k \cdots B_0 x_0$.

The problem becomes: to determine, whether $\max_{B_i \in I + \tau U} \| B_k \cdots B_0 \| \to 0 \ \ as \ \ k \to \infty$?

Answer: when the joint spectral radius (JSR) of the set $I + \tau U$ is smaller than 1.

Theorem 3 (N.Barabanov, 1988). The discrete system is stable $\Leftrightarrow$ $\hat{\rho}(I + \tau U) < 1.$

The system is stable $\Leftrightarrow$ there is $\tau > 0$ such that the discretized system is stable.

The problem is τ may be very small. It is a priory not clear which τ is enough.

Definition. The Lyapunov exponent $\sigma(U)$ is the infimum of α such that

$$\|x(t)\| \le C\, e^{\alpha t} \ , \quad t > 0$$

The system is stable $\Leftrightarrow$ $\sigma < 0$

Assume $sp(A) \subset \mathbb{R}$, $A \in U$, then

Theorem 2. (V.Protasov, R.Jungers, 2013)

For an arbitrary $\varepsilon > 0$, to distinct the cases $\sigma(U) < 0$ and $\sigma(U) > -\varepsilon$

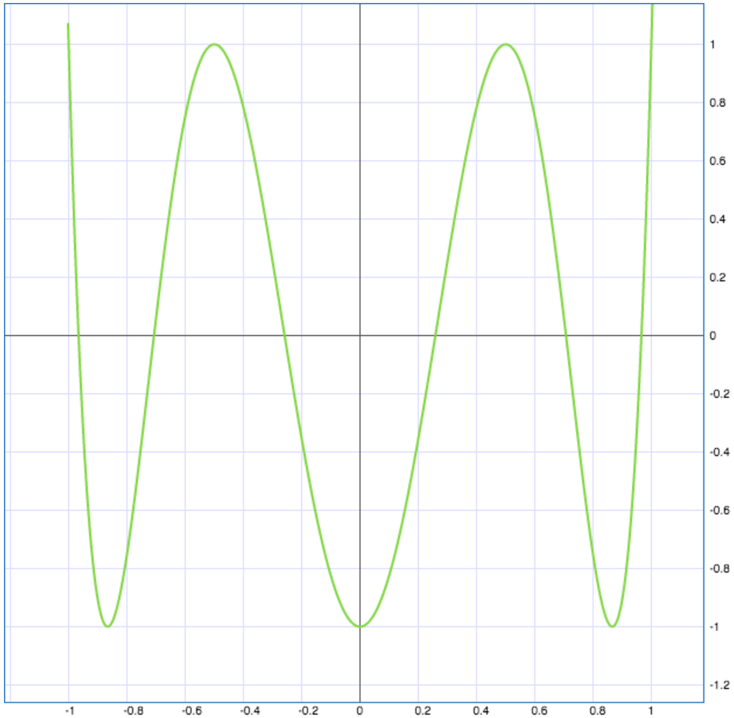
it suffices to take $\tau = \dfrac{3}{8\, r^2\, d^2}\, \varepsilon$, where $r = \max\limits_{A \in U} \rho(A)$

The idea of the proof

Theorem (A.Markov, S.Bernstein, 1889) For an algebraic polynomial of degree d, we have

$$\|p'\|_{C[-1,1]} \leq d^2 \|p\|_{C[-1,1]}$$

The equality holds only for polynomials proportional to the Chebyshev polynomial T_d



$$\|p'\|_{C[-1,1]} \leq d^2 \|p\|_{C[-1,1]}$$

This inequality can be extended to every Chebyshev system of functions.
In particular, to the sum of real exponents:

$$p(t) = \sum_{k=1}^{d} e^{-\alpha_k t}, \quad \alpha_1, \cdots, \alpha_d > 0$$

Theorem (P.B. Borwein, T. Erdélyi, 1995) For an exponentioal polynomial of degree d, we have

$$\|p'\|_{C[0,\ +\infty)} \leq c\, a\, d\, \|p\|_{C[0,+\infty)},$$

where $a = \max\{\alpha_1, \cdots, \alpha_d\}$, $c > 0$ is a constant

The sharp estimates for the constant c have been found by V.Sklyarov (2010).

We apply this inequality on exponential polynomials for the numbers

$$\alpha_k = -\lambda_k \,, \quad k = 1,....,d, \quad \text{where } \{\lambda_1,....,\lambda_d\} = \text{sp(A)}$$

However, this method is applicable to matrices with a real spectrum only!

For general complex numbers this does not work because:

Complex exponents do not form a Chebyshev system

How to solve the problem

$$p(0) \to \max$$

$$\|p\|_{C[0,+\infty)} \leq 1$$

$$p(t) = \sum_{k=1}^{d} e^{-\alpha_k t}$$

For arbitrary complex numbers $\alpha_1, \cdots, \alpha_d \in \mathbb{C}$?

Neither alternance idea nor Remez type of algorithms work here

Thank you!