Regular plane tilings and interpolation algorithms

Tatyana Zaitseva
MSU, department of Mechanics and Mathematics
Supervisor: Vladimir Protasov

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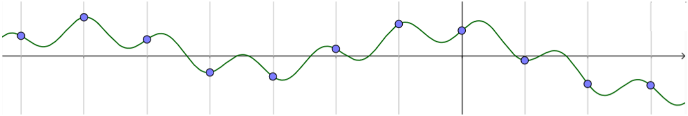
Curves through given points



$u\colon \mathbb{Z} \to \mathbb{R}$, $u \in \ell_\infty$
$D\colon u \mapsto f_u \in C(\mathbb{R})$, $D\colon \ell_\infty \to C(\mathbb{R})$:
D is linear, continuous, shift-invariant.

The method of spline interpolation is well known. There is an alternative approach working better in many practical cases.

Desirable properties of the method:

- locality
- linearity
- shift-invariance
- simplicity

Subdivision schemes

How to find f?

$f_0(k) = u(k)$, $f_0 \colon \mathbb{Z} \to \mathbb{R}$

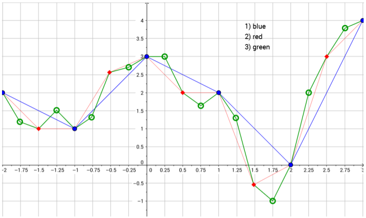
$\downarrow$

$f_1(k) = S(f_0)(2k)$, $f_1 \colon \frac{1}{2}\mathbb{Z} \to \mathbb{R}$

$\downarrow$

$f_2(k) = S(f_1)(2k)$, $f_2 \colon \frac{1}{4}\mathbb{Z} \to \mathbb{R}$

$\dots$



Subdivision operator $S \colon \ell_\infty \to \ell_\infty$ is linear, continuous, shift-invariant.

It is defined using mask $c_0, c_1, \dots, c_N$.

$$[Su](k) = \sum_{j \in \mathbb{Z}} c_{k-2j} \cdot u(j).$$

For example, $[Su](0) = c_0 u(0) + c_2 u(-1) + c_4 u(-2) + \dots$

Subdivision schemes

Scheme by De Rham (1950), A.Chaikin (1972), N.Dyn, A.Levin (1986), S.Dubuc (1986)

$a_0, a_1, \dots, a_m, b_0, \dots, b_m$ are fixed numbers. By definition, set

$$\begin{cases} f_{j+1}(q) = \sum_k a_k \cdot f_j(q - k2^{-j}), \\ f_{j+1}(q + \frac{1}{2}) = \sum_k b_k \cdot f_j(q - k2^{-j}) \end{cases}$$

Subdivision scheme is called **convergent** if $\forall f_0 \colon \mathbb{Z} \to \mathbb{R}$ $\exists g_{f_0} \in C(\mathbb{R})$ such that $\|f_j - g_{f_0}\|_{L_\infty(2^{-j}\mathbb{Z})} \to 0$

Necessary convergence condition

If $f_j \to g, j \to \infty$, then $f_j(q - k2^{-j}) \approx g(q)$. Then

$$\sum a_k = 1$$

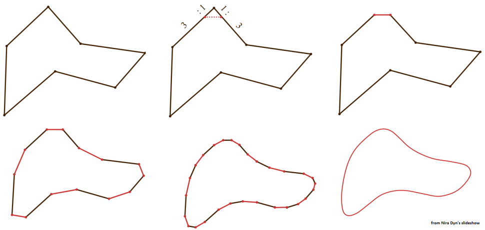
$$\sum b_k = 1$$

Let $c_{2k} = a_k, c_{2k+1} = b_k$.
$f_j(2^{-j}k) = [S^j f_0](k)$
$\|S^j f_0(k) - g(2^{-j}k)\|_\infty \to 0 \ (j \to \infty)$

$$\sum c_{2k} = \sum c_{2k+1} = 1$$

Origins

De Rham cutting corners method



$x^1(k) =$
$\alpha x^0(k) + (1-\alpha)x^0(k-1)$

$x^1(k+\frac{1}{2}) =$
$(1-\alpha)x^0(k) + \alpha x^0(k-1)$

$(a_0, a_1) = (\alpha, 1-\alpha)$

$(b_0, b_1) = (1-\alpha, \alpha)$

Oscar-winning Geri's Game(1998) used subdivision method. This makes realistic simulation of human skins and clothing possible.

Relation to refinement equations

- limit function only for one case (due to linearity and shift-invariance)
- let $\delta(k) = \delta_k$
- let the limit function of δ be g_δ.
- then $\forall f \in \ell_\infty \; g_f(t) = \sum_k g_\delta(t-k) \cdot f(k)$

Theorem 1: g_δ satisfies refinement equation:

$$\varphi(x) = \sum_k c_k \varphi(2x - k)$$

Convergence rate

Let T be the transition operator:

$$Tf = \sum_k c_k f(2t - k)$$

The refinement equation is $T\varphi = \varphi$.

$\forall g \;\; T^j g(t) = \sum_k u_k g(2^j t - k)$.

If $j = 1, u_k = c_k \Rightarrow u_k = S^j \delta$.

If the subdivision scheme converges,

$\| T^j g - \varphi(t) \| \to 0, j \to 0$.

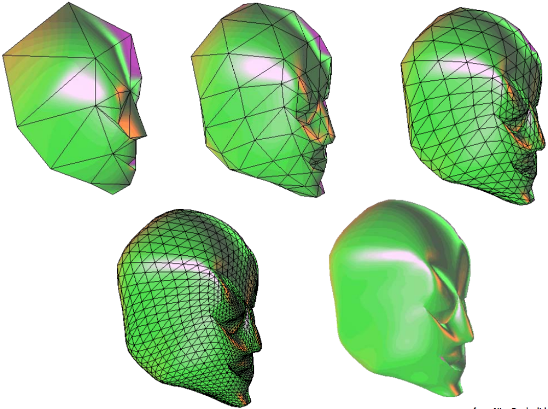
The convergence rate: $\| T^j g - \varphi(t) \|_{L_2} \approx C \cdot (\rho_2)^j$

$\rho_2(T_0, T_1) = \lim_{m \to \infty} (2^{-m} \cdot \sum_\sigma \| T_{\sigma(1)} \dots T_{\sigma(m)} \|^2)^{\frac{1}{2m}}$,

$\sigma : \{1, 2, \dots, m\} \to \{0, 1\}$

Multidimensional case

The integer lattice $\mathbb{Z}$ can be replaced by $\mathbb{Z}^d$ or by an arbitrary mesh on $\mathbb{R}^d$.



from Nira Dyn's slideshow

Multidimensional case

Binary numeral system, $m = 2$

$d_0 = 0, d_1 = 1$

$$[0,1] = \left\{ \sum_{k=1}^{\infty} 2^{-k} \cdot d_{n_k}, n_k \in \{0,1\} \right\}$$

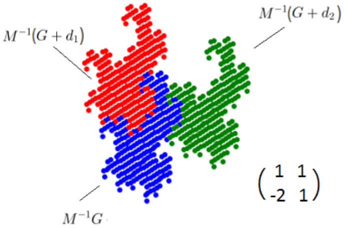
0 0.5 1

1) $G = \frac{1}{2} G \cup \frac{1}{2}(G+1)$

2) $\bigcup_{k \in \mathbb{Z}} (G+k) = \mathbb{R}$

Let M be a dilation matrix,
Ω be a set of "digits"in $\mathbb{Z}^d$

$$G = \left\{ \sum_{k=1}^{\infty} M^{-k} d_{n_k}, d_{n_k} \in \Omega \right\}$$



$$\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

Multidimensional case

$$\varphi(x) = \sum_{k\in\Omega} c_k \varphi(Mx - k)$$

Let M be a dilation matrix, Ω be a set of "digits"in $\mathbb{Z}^d$, $c_k = 1$
$\forall k \in \Omega$
$\varphi = \lambda \cdot \chi_G$, G is called a ***tile***.

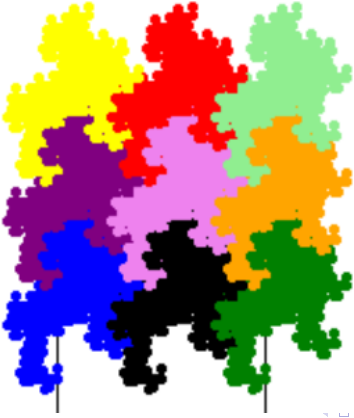
$$G = \left\{ \sum_{k=1}^{\infty} M^{-k} d_{n_k} \right\}$$

G is a compact set, it satisfies:

- $\exists$ system of 'digits' — integer vectors $d_0, d_1, ...d_{m-1}$, where $m = |\det M|$, such that $G = \bigcup_{i=1}^{m-1} M^{-1}(G + d_i)$
- $\bigcup_{k\in\mathbb{Z}^d} (G + k) = \mathbb{R}^d$ (it's multi-layer covering, term "tile"is used if there is one layer)

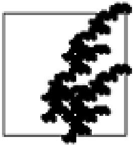
Example

If $M = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ and $d_0 = (0,0)$, $d_1 = (1,0)$, $d_2 = (1,-1)$:



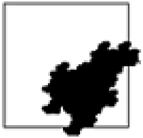
Examples

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$



d1 = (1 0)
d2 = (0 1)

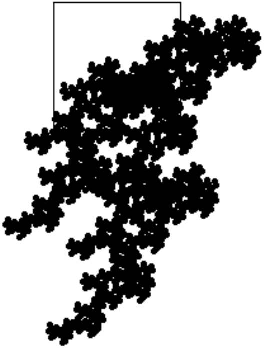
$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$



d1 = (1 0)
d2 = (1 1)

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

d1 = (3 -1)
d2 = (2 2)



Two-digit tiles and smoothness

Problems

- Classify two-digit tiles up to affine similarity in $\mathbb{R}^2$
- Find smoothness:

  Holder: $\alpha_\varphi = sup\,\{\alpha \geqslant 0 \mid \|\varphi(\cdot) - \varphi(\cdot + h)\|_2 \leqslant c \cdot h^\alpha\}$

  Sobolev:
  $s_\varphi = sup\left\{s > 0 \mid \int |\hat{\varphi}|^2(|\xi|^2 + 1)^s d\xi < \infty)\right\}$

We use the results from this article:
M. Charina and V. Yu. Protasov, *Smoothness of anisotropic wavelets, frames and subdivision schemes*, arXiv:1702.00269, 2016 year

Main result

<u>Theorem 2</u>:
For $m = 2, d = 2$, there exist three types of tiles up to affine similarity.

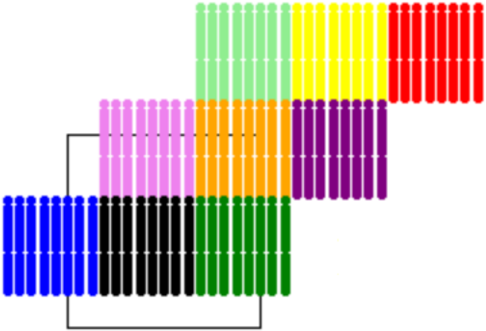
- *Rectangle*
  case $\mathrm{tr} = 0$ and $\det = \pm 2$
  smoothness: $\mathbf{0.5}$
  coeff. of conv. rate: ≈ 0.707
- *Dragon*
  case $\mathrm{tr} = \pm 2$ and $\det = 2$
  smoothness: $\approx \mathbf{0.23819}$
  coeff. of conv. rate: ≈ 0.848
- *Third*
  case $\mathrm{tr} = \pm 1$ and $\det = 2$
  smoothness: $\approx \mathbf{0.39462}$
  coeff. of conv. rate: ≈ 0.761

Rectangle

$$M = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

digits $(0,0),(1,0)$

smoothness: **0.5**

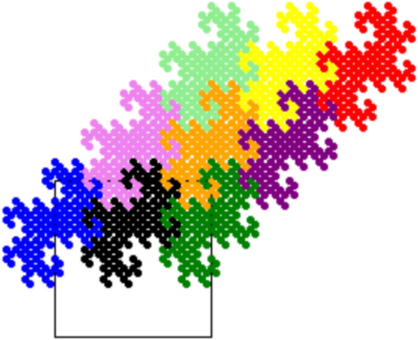


Dragon

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

digits $(0,0), (1,0)$

smoothness:
0.23819

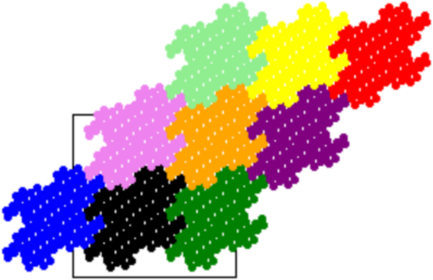


Third

$$M = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$$

digits $(0,0),(1,0)$

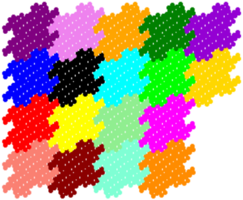
smoothness:
0.39462



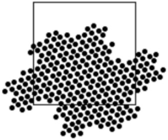
More about smoothness

- general equation: $\varphi(t) = \sum_{k \in \mathbb{Z}^d} c_k \varphi(Mx - k) \rightarrow$ G – tile
- $S = \left\{ k \in \mathbb{Z}^d \mid c_k \neq 0 \right\}$
- $G_S = \left\{ \sum_{i=1}^{\infty} M^{-i} \cdot s_i \mid s_i \in S \right\}$
- $\Omega = \{ j \in \mathbb{Z}^d \mid (G + j) \cap G \neq \varnothing \}$
- $T_d, d \in D$ $(T_d)_{ij} = c_{Mi-j+d}, i, j \in \Omega$ (in our case T_0, T_1)
- $v(x) = \varphi(x + k)_{k \in \Omega} \in \mathbb{R}^n$
- refinement equation $\leftrightarrow$ $v(x) = T_d v(Mx - d)$, $\forall x \in M^{-1}(G + d), d \in D$

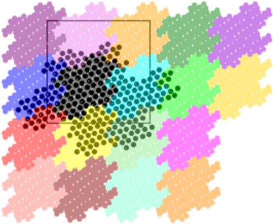
Example



digits (00) (10)



digits (00) (21)



Example

Matrices $T_{(00)}, T_{(10)}$ in $\left\{u \in \mathbb{R}^N : \sum u_i = 1\right\}$

$A_{(00)}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$A_{(10)}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\mathscr{A} X = \tfrac{1}{2}(A_0^T X A_0 + A_1^T X A_1)$$

$\rho_2 = \lambda_{max}(\mathscr{A})$ and $\rho_2 = \lambda_{max}(\tfrac{1}{2}(A_0 \otimes A_0 + A_1 \otimes A_1))$

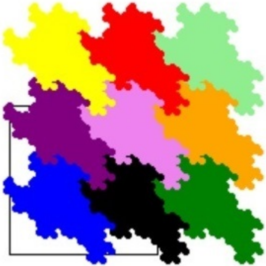
Three-digit tiles

<u>Theorem 3</u>: For $m = 3, d = 2$ there are 10 types of tiles up to affine similarity.

In this case we can also calculate smoothness.

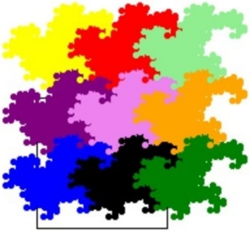
1 -1
2 2

(1, 0)
(0, 1)



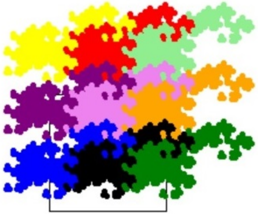
1 -2
1 1

(1, 0)
(0, 1)



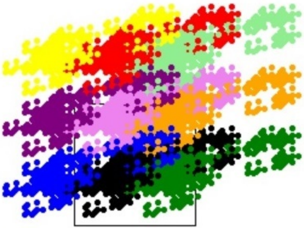
$\begin{matrix} 1 & -3 \\ 1 & 0 \end{matrix}$

(1, 0)
(0, 1)



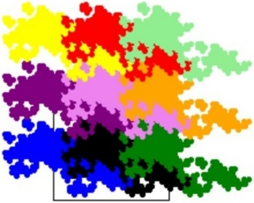
$\begin{matrix} 1 & -4 \\ 1 & -1 \end{matrix}$

(1, 0)
(0, 1)



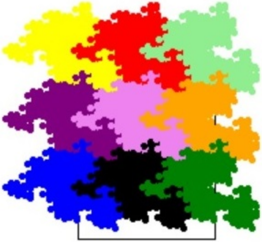
$\begin{matrix} -1 & -3 \\ 1 & 0 \end{matrix}$

(1, 0)
(1, 1)



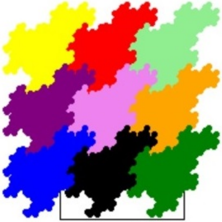
$\begin{matrix} -1 & -2 \\ 1 & -1 \end{matrix}$

(1, 0)
(1, 1)



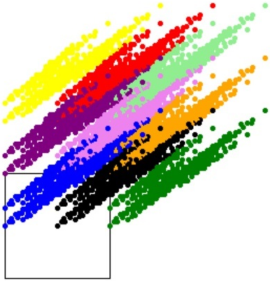
-1 -1
1 -2

(1, 0)
(1, 1)



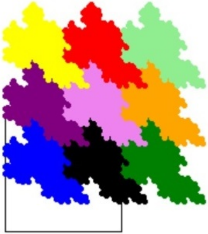
-1 3
1 0

(1, 0)
(1, 1)



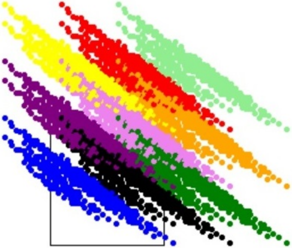
1 2
1 -1

(1, 0)
(0, 1)



1 3
1 0

(1, 0)
(0, 1)



Thank you for attention!